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LIMIT CYCLE OSCILLATION PREDICTION FOR AEROELASTIC SYSTEMS WITH CONTINUOUS NON-LINEARITIES

A. Sedaghat*, J.E. Cooper#, J. R. Wright§ and A.Y.T. Leung§
* Research Assistant # Senior Lecturer § Professor
Manchester School of Engineering
University of Manchester, Manchester M13 9PL, UK

Abstract

This paper describes an investigation into the prediction and characterisation of Limit Cycle Oscillations (LCO) occurring in non-linear aeroelastic systems. Through the use of Normal Form Theory, it is shown how it is possible to predict the amplitude of Limit Cycle Oscillations as function of two variables, the air speed and the reduced frequency. The approach is analytical and does away with the need for numerical simulation of the system. The methodology is demonstrated upon a simple two degrees-of-freedom aeroelastic wing model with cubic stiffness. The characteristics of the Limit Cycle Oscillations were determined and compared with those produced via numerical simulation. A good agreement was found for all the cases considered.

Introduction

The influence of non-linearities on modern aircraft is becoming of increasing importance[1]. These non-linearities can be structural (free-play, backlash, cubic stiffness), aerodynamic (moving shocks for transonic flows) or control (time delays, non-linear control laws) based and can result in behaviour such as Limit Cycle Oscillations (LCO). Such phenomenon cannot occur in a linear system[2] and consequently a linear analysis is unable to predict them. For example, figure 1 shows a case of Limit Cycle switching[3], where the application of an initial sine sweep results in a limit cycle response. The application of a second sine sweep causes the LCO behaviour to change to a different steady state. Figure 2 shows how the behaviour of an aeroelastic system depends upon the part of the flight envelope it is in, with some regions being susceptible to LCO, chaotic behaviour and, at higher speeds, flutter.

With the increasing use of more sophisticated technologies (e.g. smart materials, stealth) along with construction techniques (superplastic forming, diffusion bonding) that reduce the amount of inherent structural damping in the structure, the use of linear analysis techniques for aeroelastic analysis is becoming less feasible. LCO has been an increasingly important topic for a number of years[4-6] see for instance the last two proceedings of the International Forum on Aeroelasticity and Structural Dynamics (1997, 1999).

Although not desirable, LCO is essentially a fatigue problem and is not instantaneously disastrous. However, flutter is usually catastrophic and must be avoided at all costs. The ability to characterise accurately LCO and flutter, as well as predicting the flight conditions at which they occur is very important. There is a particular need to distinguish between LCO and classical flutter. Such an accurate LCO / flutter prediction capability would reduce significantly the amount of flights required in any flutter clearance test programme, with current costs being estimated at around $60k per test flight, whilst improving the safety of the clearance procedure.

There has been much work in recent years devoted towards the characterisation of non-linear aeroelastic behaviour, including LCO. The harmonic balance method is a relatively simple approach that begins to address the problem for structural non-linearities, however, due to the assumptions made in the modelling procedure, the method does not produce accurate estimates of the LCO behaviour. Other work has consisted of simulating the response of the aeroelastic system through numerical integration[2,3,7-9]. There are a few known instances of experimental verification[10]. Improved unsteady CFD[11-13] modelling, allied to the coupling of the aerodynamic and structural grids, has made significant headway towards solving the problem, particularly in the transonic region. However, there are still major problems inherent in such an approach due to the enormous computational resources required for even the simplest cases.

Research in the non-linear dynamics community has led to the development of several methods[14-21] that enable the stability boundaries of a non-linear system, whose equations of motion are known, to be defined. It is also possible to characterise the possible instabilities.

This paper describes an approach to determine the onset and characteristics of limit cycle oscillations in non-linear aeroelastic systems without the need for extensive computational simulation. Similar approaches have been developed recently [22,23]. The methodology described here is not seen as replacing the extensive CFD modelling mentioned above, but as a guide as to which parts of the flight envelope that should be investigated using sophisticated CFD methods. This work is part of a research programme aimed at developing a complete modelling and predictive capability for non-linear aircraft.

In this paper, the Normal Form theory[11] has been implemented in such a way that it can be used on non-linear aeroelastic systems (although in this work only cases where the non-linearities are continuous will be considered).

The methodology of the proposed approach is described, involving the transformation of the system equations into modal canonical form, reduction of these equations into Normal Form, and then the prediction of the instability behaviour, be it LCO or flutter. The method is demonstrated upon a simple binary aeroelastic system with structural non-linearities. Unsteady aerodynamic effects are included. The predictions are verified through comparison with numerical simulations.

Governing Equations for a Two DOF aeroelastic system

Consider a simple wing model[24], shown in Figure 3, representing a large aspect ratio wing which is here defined as a wing with rigid chordwise sections whose semi-span, s, is substantially larger than its chord, c. The wing model is fixed at its root with bending γ and torsion θ degrees of freedom. In general, the equations of motion for a general two DOF aeroelastic system including a non-linear function $F(q)$ can be expressed as follows

$$\mathbf{A}\,\ddot{q} + (\rho V\mathbf{B} + \mathbf{D})\,\dot{q} + (\rho V^2\mathbf{C} + \mathbf{E})q + F(q) = 0 \quad (1)$$

where $q=(q_1, q_2)=(\gamma, \theta)$ is the vector containing the co-ordinates components, $F(q)$ is any continuous non-linear function of q, ρ is the air density, V is the air speed. The matrices **A, B, D, C** and **E** are the inertia, aerodynamic damping, structural damping, aero-dynamic stiffness and structural stiffness matrices, respectively, given by

$$\mathbf{A} = \begin{bmatrix} I_\gamma & I_{\gamma\theta} \\ I_{\gamma\theta} & I_\theta \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \frac{cs^2a}{6} & 0 \\ -\frac{c^2s^2ea}{4} & -\frac{c^3s}{2}M_{\dot{\theta}} \end{bmatrix} \quad (2)$$

$$\mathbf{C} = \begin{bmatrix} 0 & \frac{cs^2a}{4} \\ 0 & -\frac{c^2sea}{2} \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} k_\gamma & 0 \\ 0 & k_\theta \end{bmatrix}$$

I_γ, I_θ and $I_{\gamma\theta}$ are the moments of inertia in bending, pitch and the cross product, respectively. k_γ, k_θ are the rotational stiffness in bending and torsion, respectively. The parameters c, s, e and a are the chord length, the wing semi span, the non-dimensional distance of the flexural axis from the aerodynamic centre, and the two-dimensional sectional lift curve slope, respectively. In deriving the equations of motion 1, quasi-steady aerodynamics was chosen with the exception that $M_{\dot{\theta}}$ is a non-dimensional aerodynamic torsional damping derivative. It has been shown that this term is the most critical in characterising the unsteady aeroelastic behaviour. This aerodynamic derivative is modelled in this work as a function of the reduced frequency $\nu = \omega.c / V$ such that

$$\left(-M_{\dot{\theta}}(\nu)\right)_\nu = \frac{0.491336\,\nu}{0.1996901 + \nu} \quad (3)$$

The structural damping, **D** has been ignored here as the aerodynamic damping terms dominate. Such an assumption produces conservative estimates of the flutter speed. Structural damping can be included in the model without reducing the generality of the following analysis.

Normal Form Theory

Normal Form Theory (NFT) is used to simplify analytical expression for non-linear systems. In this method, a non-linear co-ordinate transformation is employed to obtain a simple analytical expression for the transformed equations such that the qualitative behaviour of the system is evaluated without the need for solving the system of equations. In this section, the classical NFT of Poincare[15] and Brikhoff[16] is briefly discussed for a n degrees of freedom non-linear system.

Consider the non-linear ordinary differential equations in modal canonical form as follows

$$\dot{x}=Jx+f(x)=Jx+f_2(x)+\cdots+f_r(x)+O\left(|x|^{r+1}\right),\ x\in R^n \qquad (4)$$

where J is the Jordan canonical form and $f_k(x)\in H_n^k$ is the k^{th} order homogeneous polynomial in x.
In order to simplify the non-linear terms in equation 4 into their normal forms, a nearly identity non-linear co-ordinate transformation of the form

$$x=y+h_k(y),\ h_k(y)\in H_n^k,\ 2\le k\le r \qquad (5)$$

is introduced where $h_k(y)$ is the k^{th} order function of y. Substituting equation 5 into 4 and truncating up to the k^{th} order gives

$$\dot{y}=Jy+f_2(y)+\cdots+f_{k-1}(y)+f_k(y)+ad_J^k h_k(y)+O\left(|y|^{k+1}\right) \qquad (6)$$

where the function $ad_J^k h_k(y)$ denotes an adjacent operator equivalent to the function of the Lie Bracket[15,16,17,18]. The normal form equation of 6 can be finally written in the polar co-ordinate system as follows

$$\begin{bmatrix}\dot{r}\\ \dot{\theta}\end{bmatrix}=\begin{bmatrix}\Omega_1(r)\\ \Omega_2(r)\end{bmatrix} \qquad (7)$$

which can be evaluated for the stability and presence of a number of limit cycles.

It is not straightforward that to find the coefficients of normal forms using the matrix based NFT method discussed above. As an alternative, it has been shown that the averaging method[19, 20]is equivalent to the NFT method. Therefore, the problems of calculating higher order averaging equations are equivalent to the problems of calculating higher order coefficients of normal forms. Consequently, a similar approach to Leung[21] was used throughout this study.

Numerical example with cubic stiffness

Adding a cubic structural stiffness in torsion, the system equations of motion (1) can be rewritten as

$$\mathbf{A}\ddot{q}+(\rho V\mathbf{B}+\mathbf{D})\dot{q}+(\rho V^2\mathbf{C}+\mathbf{E})q+\mathbf{EN}q^3=0 \qquad (8)$$

with

$$\mathbf{EN}=\begin{bmatrix}0 & 0\\ 0 & k_\theta\end{bmatrix} \qquad (9)$$

The following numerical values for the wing model were used for this example:

$$s=10\,\text{m},\ c=3\,\text{m},\ x_{cm}=0.6c\ \text{m},\ x_f=0.5c\ \text{m},$$
$$m=200\,\text{Kg},\ a=2\pi,\ \rho=1.225\,\text{Kg/m}^3 \qquad (10)$$

where m is the mass of the wing and (x_{cm},y_{cm}) is the co-ordinates of the wing centre of mass. x_f and a are the distance of flexural axis from the wing leading edge and the two dimensional lift curve slope, respectively. ρ is the air density at sea level. The moments of inertia and stiffness coefficients were determined as

$$\begin{aligned}
I_\gamma &= \frac{ms^2}{3} = 6.667\times10^3\ (\text{Kg.m}^2)\\
I_\theta &= mc^2x^{\frac{2}{3}}{}_{cm} + m\left(x_{cm}-x_f\right)^2c^2\\
&= 2.8255\times10^3\ (\text{Kg.m}^2)\\
I_{\gamma\theta} &= m\left(x_{cm}-x_f\right)0.45sc = 8.1\times10^2\ (\text{Kg.m}^2)\\
k_\gamma &= (4\pi)^2 I_\gamma = 1.0528\times10^6\ (\text{Kg.m}^2/s^2)\\
k_\theta &= (20\pi)^2\frac{mc^2}{12} = 5.922\times10^5\ (\text{Kg.m}^2/s^2)
\end{aligned} \qquad (11)$$

A direct symbolic programming approach[25] was used to calculate the linear flutter speed of the system including unsteady effects (i.e. the system was considered with the non-linearity removed). The flutter speed and the reduced frequency were estimated as V_F=19.32 (m/s) and ν_F=2.19. The above linear flutter estimation was also compared with the standard frequency matching eigenvalue solution in figure 4, and exact agreement was achieved.

The equations of motion (8) at the flutter condition were transformed into the corresponding modal canonical form and then reduced using the Liapunov-Schmidt approach.

The reduced system is thus obtained as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}7.821\varepsilon\left(-0.0608x_1+0.0358x_2\right)^3\\ 11.976\varepsilon\left(-0.0608x_1+0.0358x_2\right)^3\end{bmatrix} \qquad (12)$$

Equation (12) was then shifted from the origin to the linear flutter condition. The calculation of any possible limit cycle oscillations can then proceed.

Limit-cycle prediction

The algebraic approach developed by Leung and Zhang[21] has been extended for application to non-linear aeroelastic problems. The results obtained were compared with Runge-Kutta numerical integration. Considering aeroelastic problems of the type considered in this paper, it was found that the reduced system equation (12) is not sufficient for obtaining normal forms. Therefore a modified version of this system was constructed such that

$$\dot{x}=Jx+f(x)=Jx+f_1(x)+f_2(x)+\cdots+f_r(x)+O\left(|x|^{r+1}\right),\ x\in R^n \quad (13)$$

where $f_1(x)=J_S x$ is the shift of the linear part of system from the origin and included into the non-linear part.

Hancock wing results for cubic stiffness

The reduced system equations (12) are shifted from the origin based on the system parameters V (the air speed) and ν (the reduced frequency) and the resultant system becomes

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}c_1x_1+c_2x_2+7.821\varepsilon\left(-0.0608x_1+0.0358x_2\right)^3\\ c_3x_1+c_4x_2+11.976\varepsilon\left(-0.0608x_1+0.0358x_2\right)^3\end{bmatrix} \quad (14)$$

where

$$\begin{bmatrix}c_1\\ c_2\\ c_3\\ c_4\end{bmatrix}=\begin{bmatrix}-0.04484+\left(-0.0005461-\dfrac{0.0005425}{0.1996901+\nu}\right)V+0.000160V^2\\ 0.031995+\left(0.00055093-\dfrac{0.0009213}{0.1996901+\nu}\right)V-0.0000943V^2\\ 0.010975+\left(0.0001815-\dfrac{0.0008307}{0.1996901+\nu}\right)V-0.0000208V^2\\ 0.0103596+\left(-0.0001831-\dfrac{0.001411}{0.1996901+\nu}\right)V+0.0000123V^2\end{bmatrix} \quad (15)$$

Here, the air speed V is taken as greater than the linear flutter speed, i.e. $V>V_F$. Applying the NFT method for obtaining normal forms up to third order, the following normal form solution is obtained

$$\begin{bmatrix}\dot{r}\\ \dot{\theta}\end{bmatrix}=\begin{bmatrix}0.5\left(c_1 r+c_4 r-0.000175r^3\right)\\ 0.125\begin{pmatrix}c_1^2-4.0c_2+c_2^2+4.0c_3+2.0c_2c_3\\ +c_3^2-2.0c_1c_4+c_4^2-0.0151r^2\end{pmatrix}\end{bmatrix} \quad (16)$$

The steady state solution for the amplitude of the limit cycles in the transformed domain can be obtained by letting the limit cycle amplitude velocity in equation (16) be zero. This leads to the expression

$$r(\nu,V)=76.6937\sqrt{c_1+c_4} \quad (17)$$

which is a general function of the air speed and the reduced frequency. Through manipulation of equations (15-17) it is possible to find expressions for the limit cycle amplitude and frequency for a given flight condition.

Figures 5 and 6 show the LCO amplitude and frequency of the Limit Cycles occurring above the linear flutter speed

The results can then be transformed back into the physical domain. As an example, implementing the steady state solution and reversing all the co-ordinate transformations, the solution to the original system (physical system) for a particular speed (ΔV=3.0 m/s) is obtained as

$$\begin{bmatrix}\gamma\\ \theta\\ \dot{\gamma}\\ \dot{\theta}\end{bmatrix}=\begin{bmatrix}\begin{pmatrix}0.0166662\,\mathrm{Cos}[17.9475\,t]+0.0008498\,\mathrm{Cos}[53.84228\,t]-0.073612\\ \mathrm{Sin}[17.9475\,t]-0.00446217\,\mathrm{Sin}[53.84238\,t]\end{pmatrix}\\ \begin{pmatrix}0.7393676\,\mathrm{Cos}[17.9475\,t]+0.019727\,\mathrm{Cos}[53.84238\,t]+0.028504\\ \mathrm{Sin}[17.9475\,t]+0.000067977\,\mathrm{Sin}[53.84238\,t]\end{pmatrix}\\ \begin{pmatrix}-0.78544\,\mathrm{Cos}[17.9475\,t]-0.01959137\,\mathrm{Cos}[53.84238\,t]-0.280468\\ \mathrm{Sin}[17.9475\,t]-0.015112\,\mathrm{Sin}[53.84238\,t]\end{pmatrix}\\ \begin{pmatrix}0.5827424\,\mathrm{Cos}[17.9475\,t]+0.09027375\,\mathrm{Cos}[53.84238\,t]-13.6730344\\ \mathrm{Sin}[17.9475\,t]-0.823233\,\mathrm{Sin}[53.84238\,t]\end{pmatrix}\end{bmatrix} \quad (18)$$

It should be noted that although the transformation contains more that one sinusoid, the response is dominated by the 17.94 (rad/sec) term. The shape of limit cycles was determined using both the Runge-Kutta numerical approach with the following set of initial conditions

$$(x_1,x_2,x_3,x_4)=(0,0,0,5.0) \text{ and } \Delta V=3.0\,(\text{m/sec})$$

and the analytical method. Figure 7 shows that there is very good agreement between the two. Note that the upper figure shows the build up to the limit cycle, whereas the lower figure shows the steady state response.

Conclusions

An analytical study has been carried out on a simple aeroelastic system containing a cubic stiffness term to demonstrate the use of Normal Form theory and the upon Liapunov-Schmidt method to predict non-linear aeroelastic behaviour. All computations were carried using symbolic computation. The approach adapted in this study is able to determine a direct analytical expression for the amplitude and frequency of the limit cycle oscillation as function of the air speed and the reduced frequency. A perfect match is obtained between the analytical and the computational results.

Further work is ongoing to apply the approach to much larger aeroelastic systems with more realistic unsteady aerodynamics including discontinuous non-linearities.

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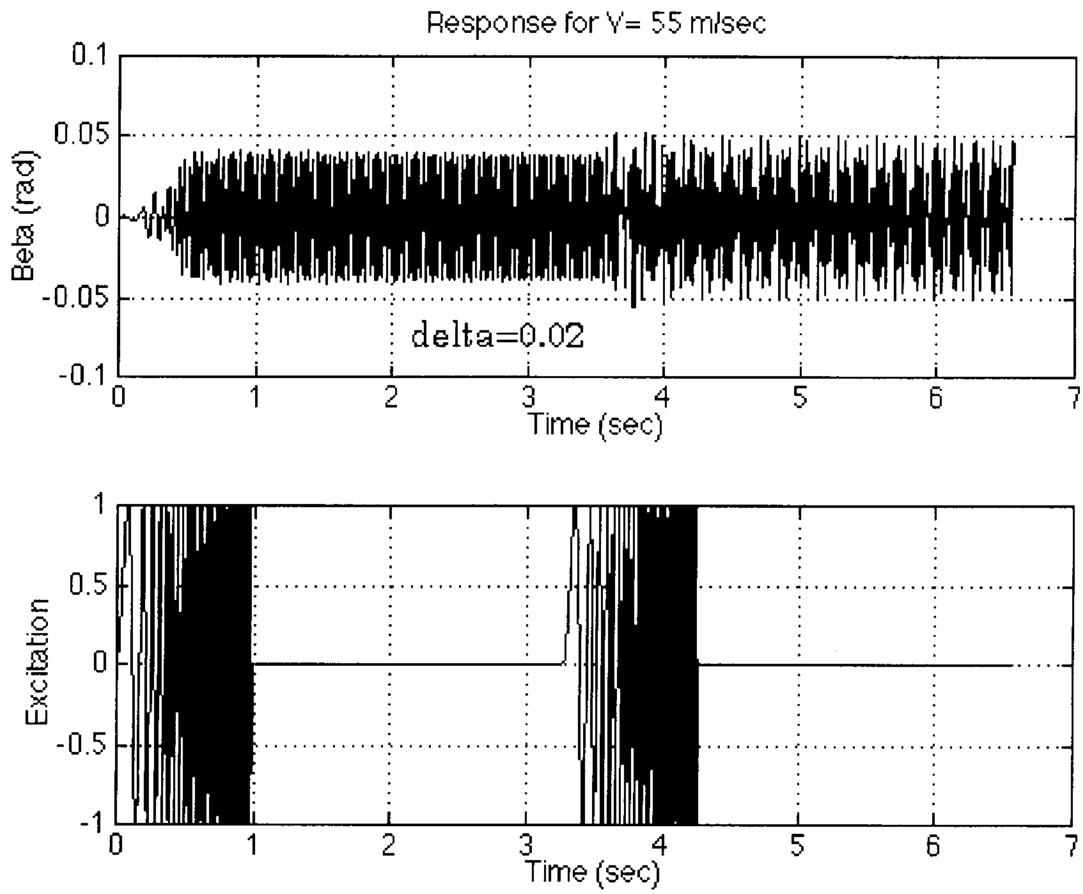

Figure 1. Example of Limit Cycle Switching for a Simple Non-Linear Aeroelastic System

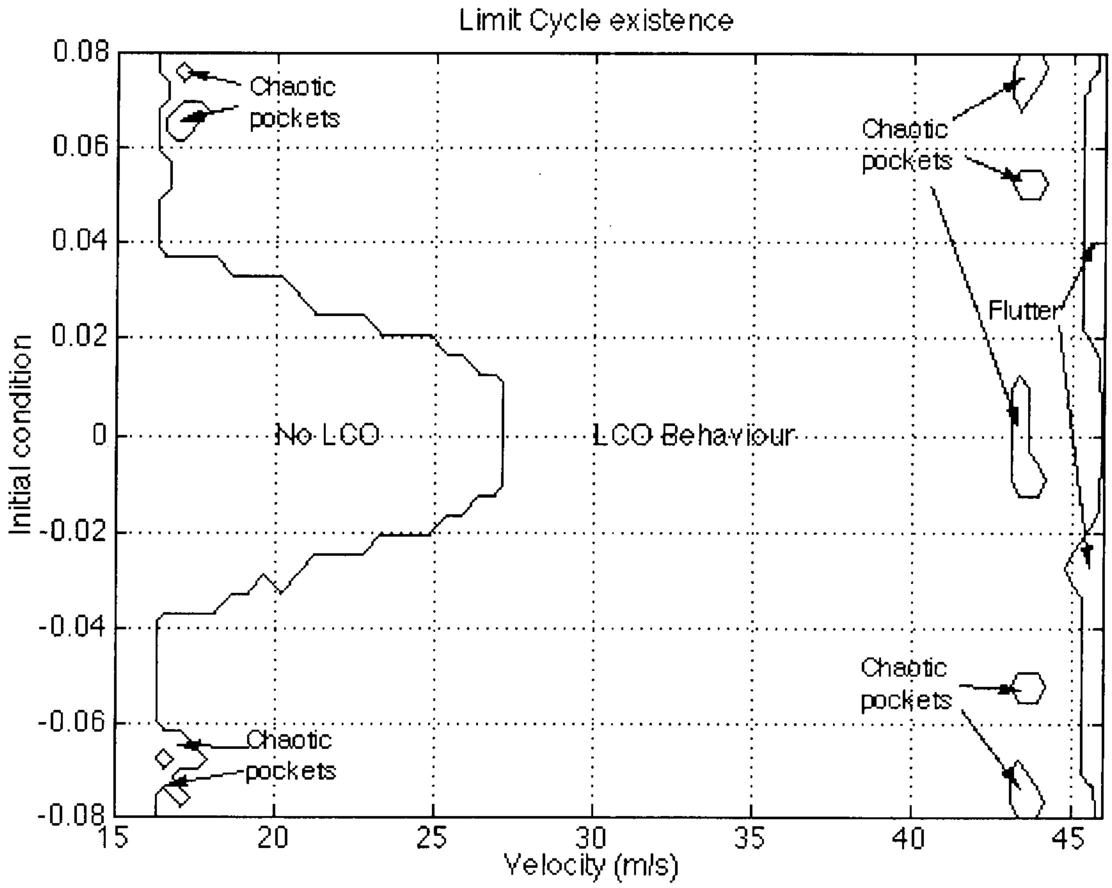

Figure 2. Example of Limit Cycle Oscillation Boundaries for a Simple Non-Linear Aeroelastic System

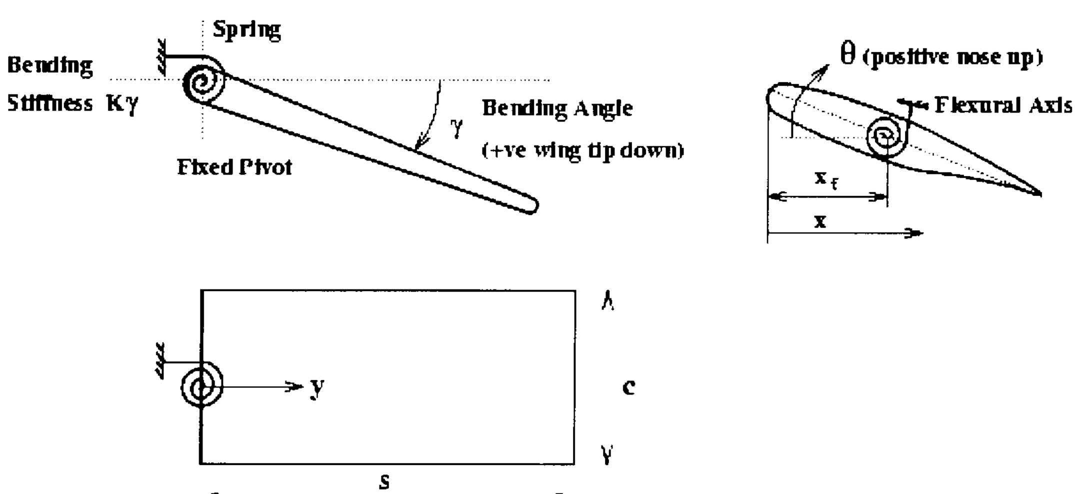

Figure 3. Rectangular wing model

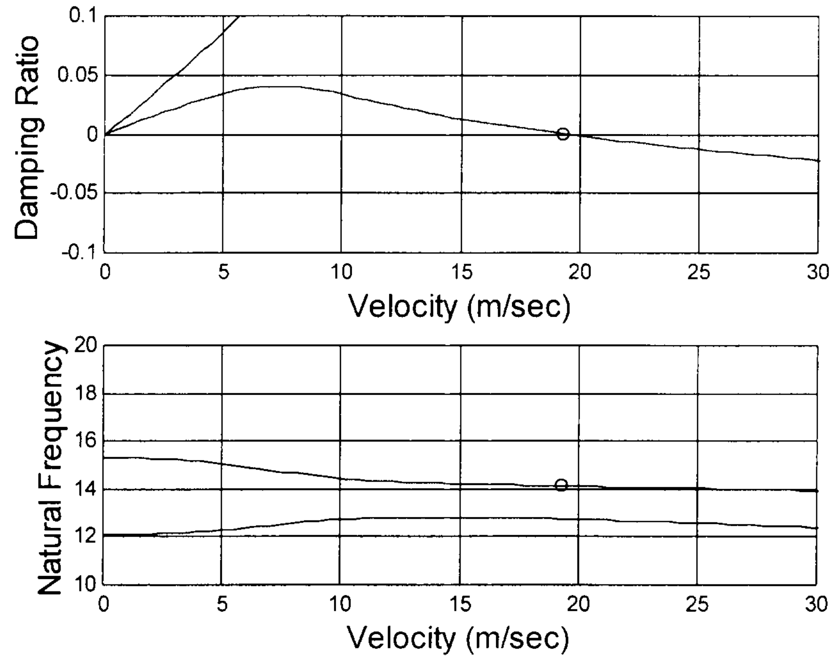

Figure 4. Frequency and Damping versus Airspeed - Linear Analysis

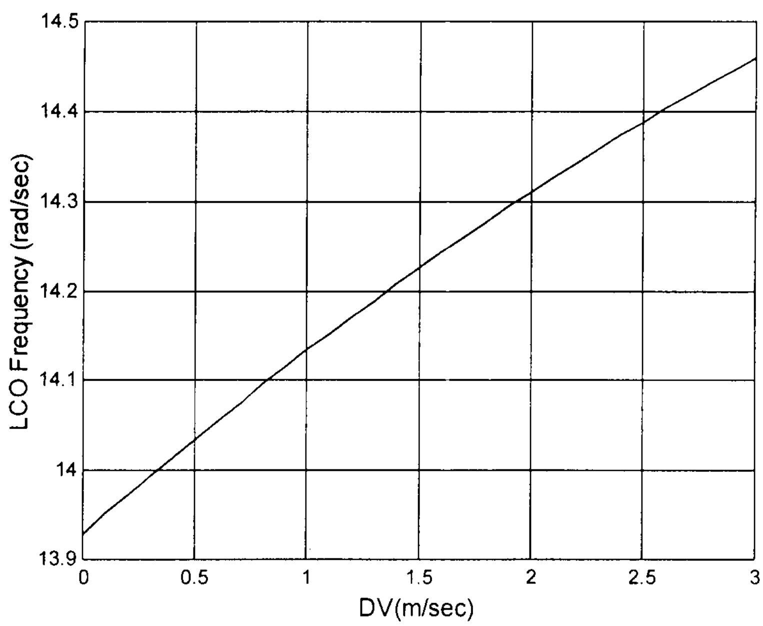

Figure 5. Variable frequency as a function of the airspeed

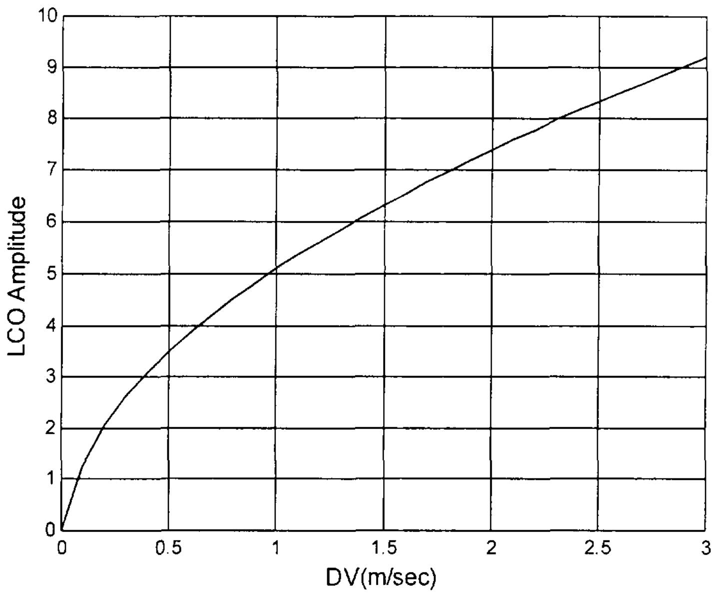

Figure 6. LCO Amplitude as a function of the airspeed

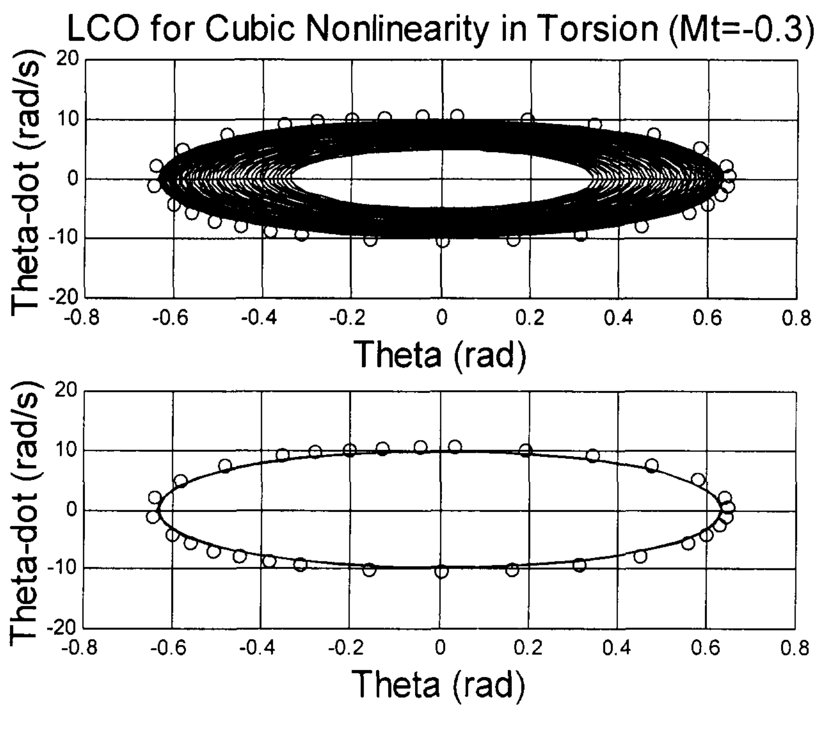

Figure 7. Comparison of the Normal Form (ooo) and the Runge-Kutta (—) limit cycle solutions.