Linear Flutter Prediction Using Symbolic Programming

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ABSTRACT: A procedure is developed to predict the speed and frequency at which flutter occurs, based upon the use of symbolic programming. The approach performs the computation in a single step and does not require the repeated calculations at various speeds required when using the classical v.g. method. By curve-fitting aeroelastic response data it is possible to find a polynomial representation of the unsteady aerodynamics. Making use of this approximation eliminates the need for iterative frequency matching schemes. The procedure is validated upon a number of simulated aeroelastic systems.

1. INTRODUCTION

Flutter is the condition when an aircraft component (e.g. wing, control surface or tail-plane) exhibits self-sustained divergent oscillations (Dowell et al.1989). The speed at which this occurs is known as the flutter speed. At speeds below the flutter speed, any initial dynamic structural vibrations will be damped, whereas at speeds above the flutter speed any initial dynamic structural disturbance will grow, leading to structural failure if not limited by aerodynamic or structural non-linearity. It is essential that all aircraft are designed such that flutter will not occur. As well as developing sophisticated structural, aerodynamic and aeroelastic mathematical models, a significant degree of testing on the ground and in the air is undertaken to demonstrate that flutter does not occur throughout the desired flight envelope (with a 20% safety margin). Ground vibration and flight flutter tests are mandatory in order to satisfy airworthiness regulations.

Having estimated the mathematical model, the flutter stability of an aircraft is usually examined by calculating the eigenvalues of the state-space form of the aeroelasticity equation at different flight conditions. In this classical method, the frequency and damping ratio of each of the complex modes derived from the eigenvalue solution are plotted against airspeed (or Mach number). A zero damping value indicates the onset of flutter. For unsteady aerodynamics, the above method is more complicated because a trial and error frequency matching scheme must be used in order to obtain the frequency of the mode that causes flutter to occur.

A simple approach to obtain exact estimates of the flutter frequency and speed for a binary flutter system, based upon the Routh-Hurwitz method, has been known for many years. However, it has not been feasible to use it on larger systems or to include frequency dependent aerodynamic forces.

In this paper, an approach is developed to calculate the flutter speed and frequency of aeroelastic systems with greater than two degrees of freedom (DOF) and also the inclusion of frequency dependent aerodynamics. A feature of the method is the use of symbolic programming to perform all the calculations. The desired quantities can be determined without the need for iteration. The method is validated using a two degree of freedom rectangular wing model, with and without unsteady aerodynamics. A further example shows the use of the approach on higher order models.

2. FLUTTER CHARACTERISTICS OF A GENERAL TWO DOF SYSTEM

The equations of motion for a general two DOF aeroelastic system can be written as

$$A\,\underline{\ddot{q}} + (\rho V B + D)\,\underline{\dot{q}} + \rho V^2 (C+E)\underline{q} = \underline{0} \qquad (1)$$

where $\underline{q}$ is the vector containing the generalised co-ordinates, ρ is the air density, V is the air (or the wing) speed. The matrices A, B, D, C and E are the mass, the aerodynamic damping, the structural damping, the aerodynamic stiffness and the structural stiffness matrices, respectively.

Using the state-space technique, the above system may be written in first order form as

$$\begin{bmatrix}\underline{\dot{q}}\\ \underline{\ddot{q}}\end{bmatrix} = \begin{bmatrix} 0 & I \\ -A^{-1}\ (\rho V^2 C + E) & -A^{-1}\ (\rho V B + D)\end{bmatrix}\begin{bmatrix}\underline{q}\\ \underline{\dot{q}}\end{bmatrix} \qquad (2)$$

and expanding equation (2) gives

$$\mathbf{\dot{z}} = \begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\\ \dot{z}_4\end{bmatrix} = \begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \ddot{q}_1\\ \ddot{q}_2\end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ f_{31} & f_{32} & f_{33} & f_{34}\\ f_{41} & f_{42} & f_{43} & f_{44}\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\\ z_4\end{bmatrix} \qquad (3)$$

where the matrix indices f_{ij} are determined based upon the mathematical model used which are functions of the airspeed V and frequency ω. The corresponding fourth order characteristic polynomial of the system can be written as

$$P(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0 \qquad (4)$$

2.1 Routh-Hurwitz Approach

Using the classical Routh-Hurwitz approach it can be shown that the critical condition for flutter to occur is

$$a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 = 0 \qquad (5)$$

from which the flutter speed can be calculated. The flutter frequency is then found as

$$\omega_F = \sqrt{\frac{a_3}{a_1}} \qquad (6)$$

2.2 Symbolic Approach

The characteristic polynomial (4) can be divided by $(\lambda^2+\omega^2)$ corresponding to a double pure imaginary roots, (± *iω*), which may be correspond to the flutter speed.

$$P(\lambda) = (\lambda^2+\omega^2)(\lambda^2 + a_1\lambda + (a_2-\omega^2)) + R \qquad (7)$$

where *R*, the remainder, is given as

$$R = \left(a_3 - a_1\omega^2\right)\lambda + \left(a_4 - (a_2-\omega^2)\omega^2\right) \qquad (8)$$

It may be noted that the terms in the remainder, R, are only a function of the airspeed V and the frequency ω. More importantly, if a physically acceptable solution for airspeed and frequency are found such that the remainder becomes zero, then the characteristic polynomial (4) has a double pure imaginary roots (± iω) corresponding to the flutter condition. This means at the flutter speed the characteristic polynomial must be divisible by $(\lambda^2+\omega_F^2)$, which yields the following equations to be solved for the flutter speed V_F and the flutter frequency $\omega_{F,}$

$$\begin{aligned} f_1(\omega_F, V_F) &= a_3 - a_1\omega_F^{\ 2} = 0 \\ f_2(\omega_F, V_F) &= a_4 - (a_2 - \omega_F^{\ 2})\omega_F^{\ 2} = 0 \end{aligned} \qquad (9)$$

This method has been programmed using a symbolic programming code.

Comparison of both approaches for the 2 DOF system shows that exactly the same solution is found using each approach

3. EXTENSION OF THE METHODOLOGY TO HIGHER-ORDER SYSTEMS

It is somewhat complex to extend the application of the Routh-Hurwitz method to higher order aeroelastic systems as the determinants that need to be solved get very complicated and there is no obvious criterion to choose for flutter, unlike the 2 DOF case. However, the extension of the symbolic approach is relatively straightforward.

For example, a three degree of freedom system yields the following characteristic polynomial

$$P(\lambda) = \lambda^6 + a_1\lambda^5 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda + a_6 \qquad (10)$$

The 6th order polynomial may be similarly divided by $(\lambda^2+\omega_F^2)$ which yields the following remainder

$$\begin{aligned} R = &\left(a_5 - \omega^2{}_F\left(a_3 - a_1\omega^2{}_F\right)\right)\lambda \\ &+ \left(a_6 - \omega^2{}_F\left(a_4 - \omega^2{}_F\left(a_2 - \omega^2{}_F\right)\right)\right) \end{aligned} \qquad (11)$$

and hence the following set of polynomial equations to be solved for flutter speed and frequency

$$f_1(\omega_F, V_F) = a_5 - \omega^2{}_F\left(a_3 - a_1\omega^2{}_F\right) = 0$$
$$f_2(\omega_F, V_F) = a_6 - \omega^2{}_F\left(a_4 - \omega^2{}_F\left(a_2 - \omega^2{}_F\right)\right) = 0 \quad (12)$$

In general, for a polynomial of order *2n*, $P_{2n}(\lambda)$, the remainder of division by $(\lambda^2+\omega_F^2)$ can be written as

$$R = \left(a_{2n-1} - \omega^2{}_F\left(a_{2n-3} - \omega^2{}_F \cdots \left(a_3 - a_1\omega^2{}_F\right)\right)\ldots\right)_{\frac{2n-2}{2}}\lambda$$
$$+\left(a_{2n} - \omega^2{}_F\left(a_{2n-2} - \omega^2{}_F \ldots \left(a_2 - \omega^2{}_F\right)\right)\ldots\right)_{\frac{2n}{2}} \quad (13)$$

where the bracket subscripts represent the number of brackets.

4. HANCOCK RECTANGULAR WING MODEL

Consider a rigid wing of constant chord (shown in Fig. 1) pivoted at its root in bending γ and torsion θ such that there is no stiffness coupling between the motions (Hancock et al. 1985).

The equations of motion for the wing model is given by (1) where q =$(\gamma \;\; \theta)^T$ is the vector of generalised co-ordinates. For the simple wing model, it can be shown that

$$\mathbf{A} = \begin{bmatrix} I_\gamma & I_{\gamma\theta} \\ I_{\gamma\theta} & I_\theta \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \frac{cs^2a}{6} & 0 \\ -\frac{c^2s^2ea}{4} & -\frac{c^3s}{2}M_{\dot\theta} \end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix} 0 & \frac{cs^2a}{4} \\ 0 & -\frac{c^2sea}{2} \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} k_\gamma & 0 \\ 0 & k_\theta \end{bmatrix} \quad (14)$$

in which I_γ, I_θ, and $I_{\gamma\theta}$ are the moments of inertia in bending, in pitch and their product, respectively and k_γ, k_θ are rotational stiffness in bending and torsion, respectively. The parameters *c, s, e* and *a* are the chord length, the wing semi-span, the non-dimensional distance of flexural axis from aerodynamic centre, and the two-dimensional sectional lift curve slope, respectively. In deriving equation (14), quasi-steady aerodynamics was used with the inclusion of the $M_{\dot\theta}$ term, a non-dimensional aerodynamic torsional damping derivative which is introduced to represent an unsteady aerodynamics effect. The structural damping, D, has been ignored here; however, it can be included without reducing the generality of the above analysis.

5. NUMERICAL EXAMPLES

Consider the above rectangular wing with the following characteristics

$$s = 10\text{ m},\; c = 3\text{ m},\; x_{cm} = 0.6c,\; x_f = 0.5c,\; m = 200\text{ Kg},$$
$$a = 2\pi,\; \rho = 1.225\text{ Kg/m}^3 \quad (15)$$

where m is the mass of the wing and (x_{cm}, y_{cm}) is coordinate components of wing centre of mass. x_f and a are the distance of flexural axis from wing leading edge and two dimensional lift curve slope, respectively. ρ is air density at zero altitude. The parameters *e*, $I_{\gamma\theta}$ and $M_{\dot\theta}$ were varied through the following examples accordingly for different flutter conditions.

5.1. *Example with Constant Aerodynamics*

Consider the Hancock wing model with the wing mass balanced (i.e. $I_{\gamma\theta}$=0), the unsteady aerodynamic damping term set as $M_{\dot\theta} = -0.1$ and *e=0.25* . Applying the approach developed in this paper leads to the expressions

$$f_1(\omega_F, V_F) = 120.909 - 0.00442\;V_F^2 - 0.583\;\omega_F^2 = 0$$
$$f_2(\omega_F, V_F) = 33095.9 - 4.839\;V_F^2 - 367.496\;\omega_F^2 + 0.0273\;V_F^2\omega_F^2 + \omega_F^4 = 0 \quad (16)$$

from which it was found that

ω_F = 14.391 rad/s and V_F = 7.523 m/s

Figure 2 shows the corresponding vg plot for this case. The circles indicate the results found using the symbolic approach. It can be seen that there is an exact agreement between the symbolic and iterative approaches.

5.2 *Example with Frequency Dependent Aerodynamics*

The next example is an extension of the previous example, where again the wing is mass balanced about the flexural axis (i.e. $I_{\gamma\theta}$ =0) but now the damping term $M_{\dot\theta}$ is assumed to vary depending upon the value of the frequency parameter υ. Based upon the values of υ given in (Hancock et al 1985) $M_{\dot\theta}$ was curve-fitted to obtain the expression

$$\left(-M_{\dot{\theta}}(\nu)\right)\nu = \frac{0.491336\nu}{0.1996901+\nu} \qquad (17)$$

This simple function can be easily inserted into the symbolic approach, and the flutter speed and frequency are found in a single step. A residual of between ±5 % was obtained between the function and the data. This could be reduced if a more complicated curve-fitting expression were used.

The symbolic equations are now found as

$$f_1(\nu_F, V_F) = 120.985 + \frac{4.541 - 0.0641(0.223+\nu_F)(0.05+0.0264\nu_F+\nu_F^2)V_F^2}{0.1997+\nu_F} = 0$$

$$f_2(\nu_F, V_F) = 33095.9 - 3.9296V_F^2 - 40.8329\nu_F^2 V_F^2 + 0.00276\nu_F^2 V_F^4 + \frac{\nu_F^4 V_F^4}{81} - \frac{0.00184\nu_F^2 V_F^4}{0.1997+\nu_F} = 0 \qquad (18)$$

which were solved to give estimates of ω_F = 14.45 rad/s and V_F = 5.979 m/s.

Comparison with the vg plot including frequency matching, as shown in figure 3, again indicates that there is an exact agreement.

5.3. *Example with a 3 and 5 DOF System*

As a final example, a 3 and 5 DOF aeroelastic systems were created that consisted of 1 bending and 2 torsion modes, and 2 bending and 3 torsion modes of a rectangular wing respectively. As before, strip theory with the Hancock modification to include unsteady aerodynamics was employed.

When the approach developed in this paper was applied to both systems, it was found that for the 3 DOF case ω_F = 99.26 rad/s and V_F = 86.80 m/s, and ω_F = 99.71 rad/s and V_F = 86.56 m/s for the 5 DOF case were predicted based upon the above approach. Figures 4 and 5 shows the corresponding vg plots. The flutter speed and frequency are estimated exactly in both cases.

6. CONCLUSIONS

A new approach has been introduced for calculating the flutter condition of linear multi DOF aeroelastic systems. By using an approach based upon symbolic computation, the method can be employed for any higher order aeroelastic system. The need for frequency matching is also eliminated if the unsteady aerodynamics is curve-fitted as a function of frequency parameter. This approach has advantages over more traditional methods that calculate a large number of eigenvalues at different airspeeds until the approximate flutter speed and frequency are obtained. The method has been demonstrated successfully on a number of simple aeroelastic systems.

7. ACKNOWLEDGMENTS

This work was supported through EPSRC Grant GR/L95175.

8. REFERENCES

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G.J. Hancock, J.R. Wright and A. Simpson, 1985, On the teaching of the principles of wing flexure-torsion flutter, *The Aeronautical Journal*, 285-305.

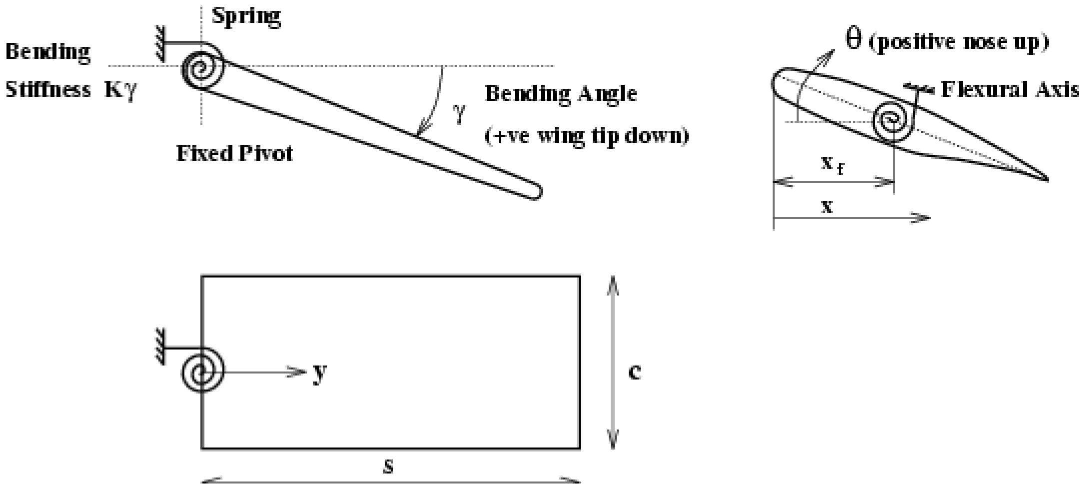

Figure.1 Schematic of the rectangular wing model

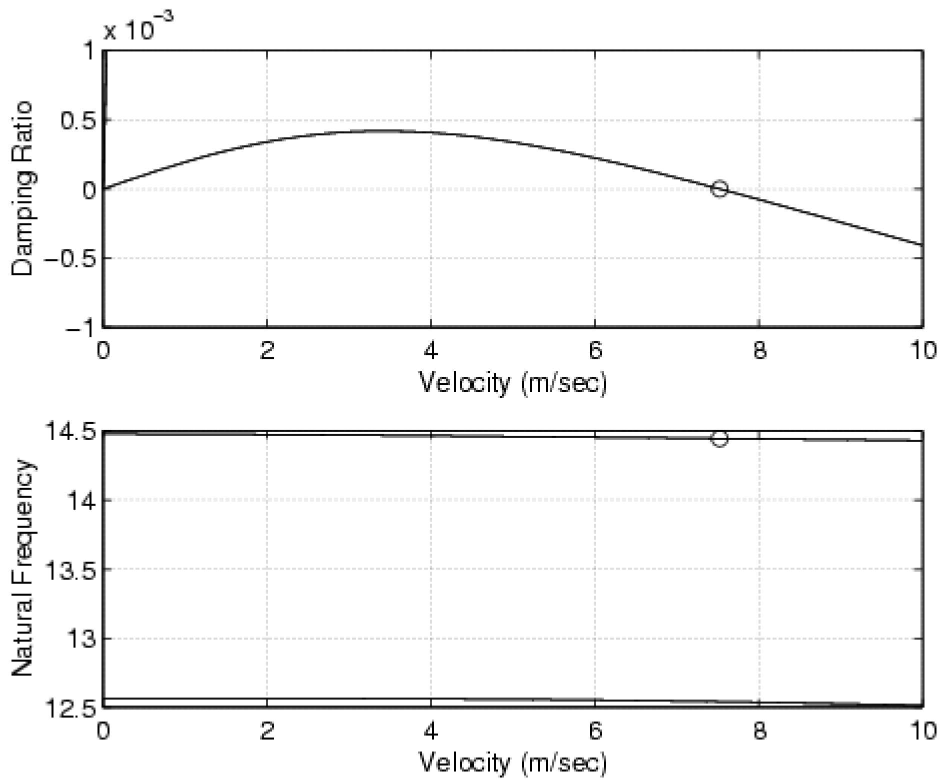

Figure 2. Frequency and Damping Trends for Constant Aerodynamics Case.

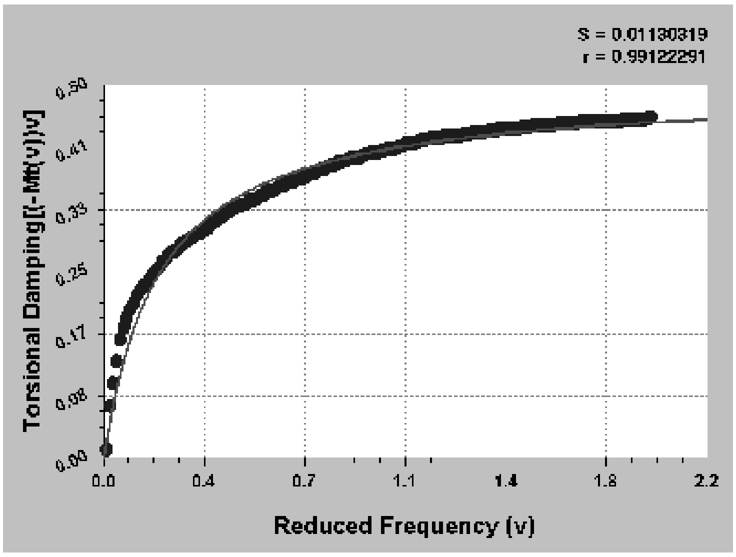

Figure 3. Curve-fit of $M_{\dot{\theta}}$

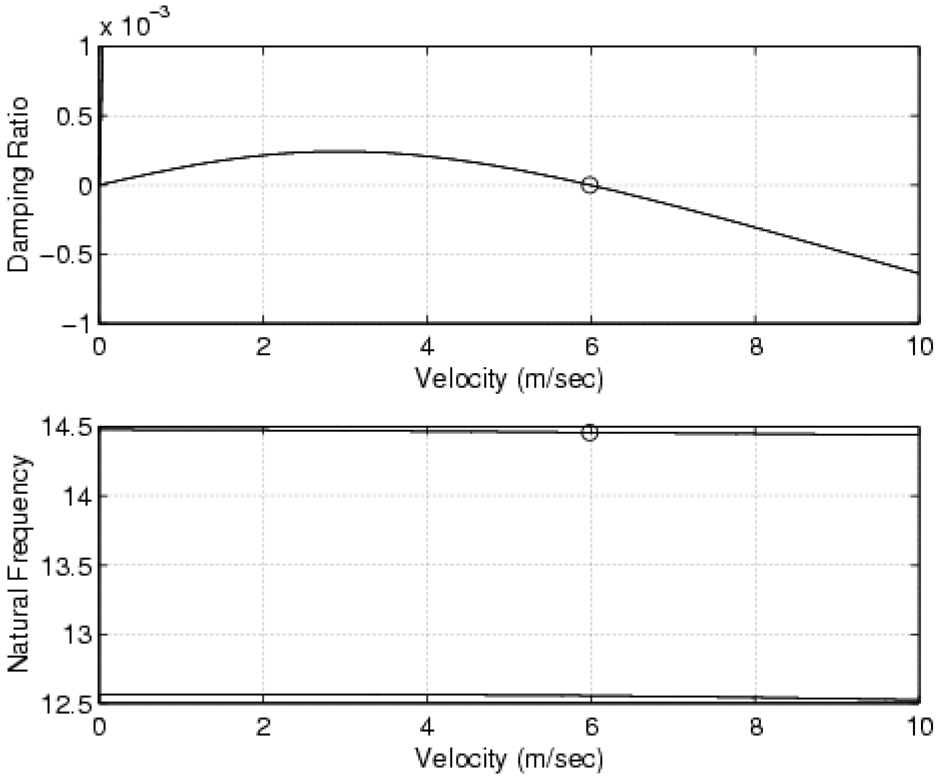

Figure 4. VG Plot of Unsteady Binary System

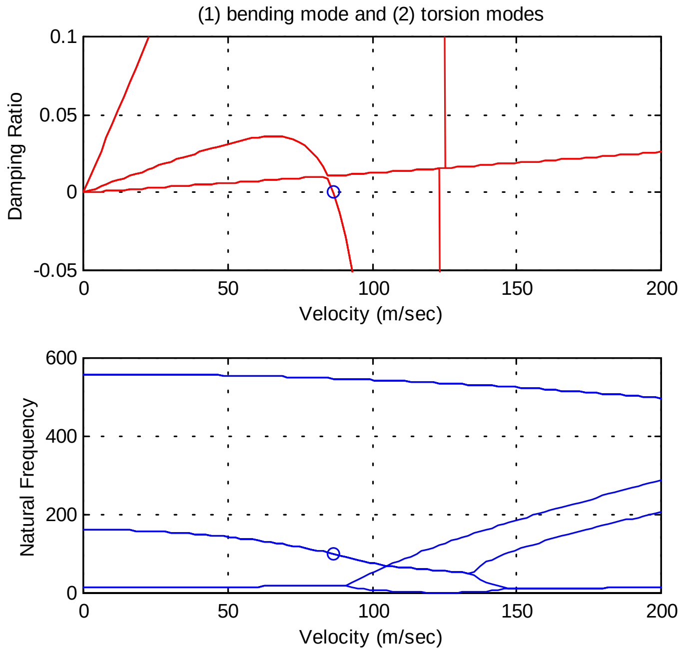

Figure 5. VG Plot of 3 DOF System

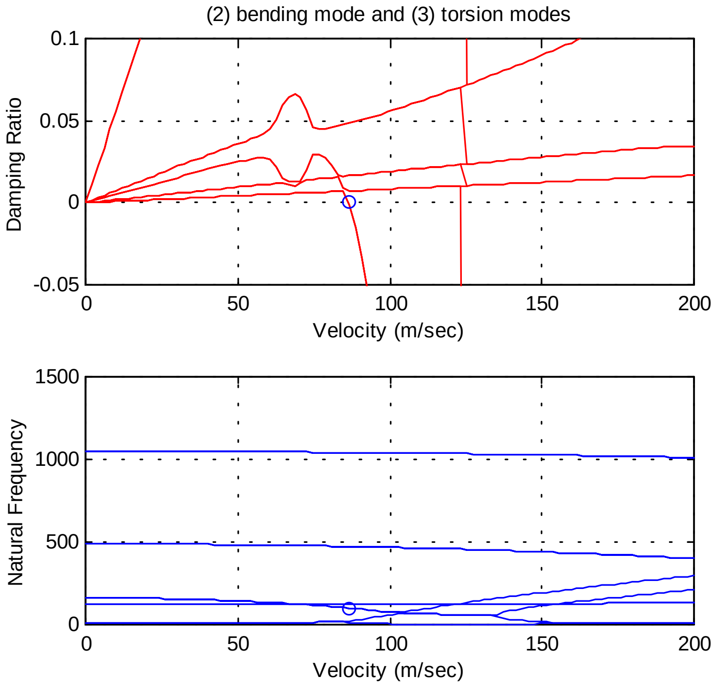

Figure 6. VG diagram of 5 DOF System