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NORMAL FORM SOLUTION OF REDUCED ORDER OSCILLATING SYSTEMS

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Abstract
This paper describes a preliminary investigation into the use of normal form theory for modelling large non-linear dynamical systems. Limit cycle oscillations were determined for a simple two-degrees-of-freedom double pendulum system. The double pendulum system was reduced into its centre manifold before applying normal form computations. Normal forms were obtained using a period averaging method which is applicable to non-autonomous systems, more advantageous than the classical methods. Good agreement was obtained between the predicted results from the normal form theory and numerical simulations of the original system.

Keywords: Normal Form Theory, Period Averaging, Limit Cycle Oscillations.

Introduction
A periodic solution of a dynamical system is called a limit cycle if there are no other periodic solutions sufficiently close to it. In other words, a limit cycle is an isolated periodic solution and corresponds to an isolated closed orbit in the state space [1]. Every trajectory initiated near a limit cycle approaches it as $t \to \infty$.

The prediction of limit cycle oscillations (LCO) has been cited for a range of simple non-linear dynamical systems [2-15] using normal form theory (NFT). The NFT is used to simplify analytical expression for non-linear systems [3-4]. In this method, a non-linear co-ordinate transformation is employed to obtain a simple analytical expression for the transformed equations such that the qualitative behaviour of the system is evaluated without the need for solving the system of equations. The classical approach of Poincare [16] and Brikhoff [17] suffers from evaluating large matrices to obtain normal forms. Liu [18] and Grzedzinski [19] have applied the centre manifold theory to reduce the number of differential equations before computing normal forms. Zhang [7] has calculated normal forms through a period averaging method that has advantages for determining non-autonomous systems (or forced vibration systems).

In this paper, the method developed by Zhang [10] was adopted for solving LCOs for a two-degrees-of-freedom double pendulum system. The non-linear system is reduced into its critical modes (or centre manifold) which may correspond to one or two single DOF systems. The reduced systems exhibit exactly the same behaviour as the full system at the corresponding modes and higher order normal forms can be obtained with less computational effort. The methodology of the approach is described, involving the transformation of the system equations into modal canonical form, reduction of these equations into normal forms, and then the prediction of the instability behaviour, here LCO. The predictions are verified through comparison with numerical simulations.

Normal Form Theory
In this section, the classical NFT of Poincare [16] and Brikhoff [17] are briefly discussed for a non-linear system with multi degrees of freedom. Consider the non-linear ordinary differential equation,

$$\dot{u} = g(u), \quad g \in C^r\left(R^n\right), \; u \in R^n \tag{1}$$

where u is a time dependent vector with n elements and the function g is differentiable up to an integer order r. The differentiation with respect to time t is shown by a dot. If equation (1) satisfies $g(u_0)=0$ at $u=u_0$, then it can be shifted to the origin by the variable exchange $v=u-u_0$ as follows

$$\dot{v}=g(v+u_0)=H(v)=H_1v+H_2(v) \tag{2}$$

Here $H(v)$ is split into the linear part, H_l, and the non-linear part H_2. The linear part $H_1=D_vH(v=0)$ is the Jacobian of $H(v)$ evaluated at $v=0$, where D_v is a differential operator with respect to v, and the non-linear part $H_2=H(v)-H_1v$ is at least a quadratic function of v. The matrix H_l is further transformed into Jordan canonical form using the canonical matrix Q, i.e. $v=Qx$, where Q is the right eigenvectors of H_l if H_l is non-defective and is generalized (principal) vectors if H_l is defective. Applying this transformation yields

$$\dot{x}=Jx+f(x)=Jx+f_2(x)+\cdots+f_r(x)+O\left(|x|^{r+1}\right),\quad x\in R^n \tag{3}$$

, where J is the Jordan canonical form $J=Q^{-1}H_1Q,\ \ f(x)=Q^{-1}H_2(Qx)$ and $f_k(x)\in H_n^k$ is the k^{th} order homogeneous polynomial in x.

In order to simplify the non-linear terms in equation (3) into their normal forms, a nearly identity non-linear co-ordinate transformation of the form

$$x=y+h_k(y),\quad h_k(y)\in H_n^k,\quad 2\le k\le r \tag{4}$$

is introduced where $h_k(y)$ is the k^{th} order function of y.

Substituting equation (4) into (3) and truncating up to the k^{th} order gives

$$\dot{y}=Jy+f_2(y)+\cdots+f_{k-1}(y)+f_k(y)+ad_J^kh_k(y)+O\left(|y|^{k+1}\right) \tag{5}$$

where the function $ad_J^kh_k(y)$ denotes an adjacent operator equivalent to the function of the Lie Bracket

$$ad_J^kh_k(y)\equiv L_J(H_k)=Jh_k(y)-D_yh_k(y)Jy \tag{6}$$

, where $D_yh_k(y)$ is the Jacobian matrix of $h_k(y)$. Note that the term, $f_j\ (2\le j\le k)$, remains unchanged under transformation (4). If a polynomial $h_k(y)$ can be found so that

$$f_k(y)+Jh_k(y)-D_yh_k(y)Jy=0=G_k(y) \tag{7}$$

, then the k^{th} order homogeneous polynomial terms in (5) are completely eliminated. Otherwise, a complementary space $G_k(y)$ at $L_J(H_k)$ must be found such that only terms of order k in $G_k(y)$ remain in the resultant expression (see [3-4, 7]).

Period Averaging Method

It is computationally exhaustive to find the coefficients of normal forms using the matrix approach explained in the previous section. An alternative faster approach is the averaging method [5-6] which is equivalent to the NFT method. Thus, the problem of calculating higher order coefficients of normal forms is equivalent to the problem of calculating higher order averaging equations. Since the averaging method is applied to non-autonomous systems, a co-ordinate transformation is chosen such that an autonomous system is obtained through a time integrating procedure.

In this approach, the following non-linear ordinary differential equation

$$\dot{x}=Jx+\varepsilon f(x,\varepsilon),\quad x\in\Omega\in R^n \tag{8}$$

is transformed to

$$\dot{y}=\varepsilon e^{-tJ}f\left(e^{tJ}y,\varepsilon\right)=\varepsilon g(y,t,\varepsilon) \tag{9}$$

by using the transformation function

$$x=e^{tJ}y \quad \text{and} \quad \dot{x}=Je^{tJ}y+e^{tJ}\dot{y} \tag{10}$$

, where $0<|x|<<1,\ \ f\in C^{r+1}$ and $f(0,\varepsilon)=0$. Here, Ω is a domain which contains the origin and invariant under Γ, $\Gamma x\in\Omega$ if $x\in\Omega$. Note that the equation (9) is explicitly dependent on time while the original equation (8) is not.

The period averaged normal forms of equation (9) can be constructed using the following variable exchange

$$y=\zeta+\sum_{l=1}^{m}\varepsilon^lh_l(\zeta,t) \tag{11}$$

, which transforms equations (9) to normal forms up to order m as follows

$$\dot{\zeta}=\sum_{k=1}^{m}\varepsilon^kf_k^0(\zeta)+O\left(\varepsilon^{m+1}\right) \tag{12}$$

, where the geometrical transformations $h_k(\zeta,t)$ in equation (11) are determined from

$$h_k(\zeta,t)=\frac{1}{T}\int_0^T\tau\left[g_k(\zeta,\tau+t)-f_k^0\right]d\tau \tag{13}$$

, and the normal forms $f_k^0(\zeta)$ are given by

$$f_k^0(\zeta)=\frac{1}{T}\int_0^Tg_k(\zeta,\tau)d\tau \tag{14}$$

, where the function $g_k(\zeta,t)$ is determined by

$$g_k(\zeta,t) = \frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial \varepsilon^{k-1}} g\left(\left(\zeta + \sum_{l=1}^{k-1} \varepsilon^l h_l(\zeta,t)\right), t, \varepsilon\right)_{\varepsilon=0} - \sum_{l=1}^{k-1} h'_{k-1}(\zeta,t) f_l^0(\zeta) \quad (15)$$

, in which a prime denotes differentiation with respect to ζ . More details on deriving the above relationships can be found in [24].

Double Pendulum System

The double pendulum system shown in Figure 1 consists of two rigid weightless links of equal length *l* which carry two concentrated mass 2m and m, respectively.

A follower force P is applied to this system. The equations of motion can be obtained for this system using the Lagrange equation [20-21]

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \left(\frac{\partial D}{\partial \dot{q}_i}\right) - \left(\frac{\partial T}{\partial q_i}\right) + \left(\frac{\partial V}{\partial q_i}\right) = Q_i, \quad i = 1,2 \quad (16)$$

where T is the kinetic energy; D is the dissipation function; V is the potential function; and Q_i = ′(*dW*)/′(*dq_i*) is the i^{th} generalized force with *dW* being the work done by the force Q_i moving through the incremental displacement dq_i. Considering the above system, the kinetic energy T of the system is given by reference [22] as follows

$$T = \frac{ml^2}{2\Omega^2}\left[3\theta_1^2 + \theta_2^2 + 2\theta_1\theta_2 \cos(\theta_1 - \theta_2)\right] \quad (17)$$

, where θ_1 and θ_2 are generalized co-ordinates specified the configuration of the system. The potential energy for the three linear spring k_1, k_2 and k_3 is given by [22]

$$V = \frac{1}{2}\begin{bmatrix} \left(k_1 + k_2 + k_3 l^2\right)\theta_1^2 + 2\left(k_3 l^2 - k_2\right)\theta_1\theta_2 \\ + \left(k_2 + k_3 l^2\right)\theta_2^2 \end{bmatrix} - \frac{1}{6}k_3 l^2 (\theta_1 + \theta_2)\left(\theta_1^3 + \theta_2^3\right) \quad (18)$$

With the aid of the Lagrange equations, one can drive a set of first order differential equations up to third order terms, see reference [22], as follows

$$\begin{aligned}
\frac{dz_1}{dt'} &= z_2, \\
\frac{dz_2}{dt'} &= -\frac{1}{2}\left(f_1 + 2f_2 - \eta\right)z_1 - \frac{3}{2} f_4 z_2 \\
&\quad + \frac{1}{2}\left(2f_2 - \eta\right)z_3 + f_4 z_4 \\
&\quad + \frac{1}{12}\left(3f_1 + 9f_2 + 2f_3 - 4\eta\right)z_1^3 \\
&\quad - \frac{1}{4}\left(2f_1 + 9f_2 - 2f_3 - 4\eta\right)z_1^2 z_3 \\
&\quad + \frac{1}{4}\left(f_1 + 9f_2 - 4\eta\right)z_1 z_3^2 \\
&\quad - \frac{1}{6}\left(3f_2 + 3f_3 - 2\eta\right)z_3^3 \\
&\quad + \frac{1}{4} f_3 \left(3z_2 - 2z_4\right)\left(z_1 - z_3\right)^2 \\
&\quad - z_2 z_4 \left(z_1 - z_3\right), \\
\frac{dz_3}{dt'} &= z_4, \\
\frac{dz_4}{dt'} &= +\frac{1}{2}\left(f_1 + 4f_2 - 2f_3 - \eta\right)z_1 + \frac{5}{2} f_4 z_2 \\
&\quad - \frac{1}{2}\left(4f_2 + 2f_3 - \eta\right)z_3 - 2 f_4 z_4 \\
&\quad - \frac{1}{12}\left(6f_1 + 15f_2 - 2f_3 - 7\eta\right)z_1^3 \\
&\quad + \frac{1}{4}\left(4f_1 + 15f_2 - f_3 - 7\eta\right)z_1^2 z_3 \\
&\quad - \frac{1}{4}\left(2f_1 + 15f_2 - 2f_3 - 7\eta\right)z_1 z_3^2 \\
&\quad + \frac{1}{12}\left(15f_2 + 14f_3 - 7\eta\right)z_3^3 \\
&\quad - \frac{1}{4} f_4 \left(5z_2 - 4z_4\right)\left(z_1 - z_3\right)^2
\end{aligned} \quad (19)$$

, where the state variables z_i are defined as

$$z_1 = \theta_1, \quad z_2 = \dot{\theta}_1, \quad z_3 = \theta_2, \quad z_4 = \dot{\theta}_2 \quad (20)$$

, and the non-dimensional quantities f_i and η are given by

$$f_1 = \frac{k_1\Omega^2}{ml^2}, \quad f_2 = \frac{k_2\Omega^2}{ml^2}, \quad f_3 = \frac{k_3\Omega^2}{m}, \quad f_4 = \frac{d\Omega^2}{ml^2}, \quad \eta = \frac{P\Omega^2}{ml} \quad (21)$$

, where f_i (i=1, 2, 3, and 4) is employed due to physical constraints and η is a system indicator parameter.

The system of equation (19) can be rewritten as

$$\dot{z} = Az + g(z) \quad (22)$$

, where Az is the linear part and $g(z)$ is the non-linear part. The Jacobian matrix A is evaluated at $z=0$ as follows

$$A=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{2}\left(f_1+2f_2-\eta\right) & -\frac{3}{2}f_4 & \frac{1}{2}\left(2f_2-\eta\right) & f_4 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2}\left(f_1+4f_2-2f_3-\eta\right) & \frac{5}{2}f_4 & -\frac{1}{2}\left(4f_2+2f_3-\eta\right) & -2f_4 \end{pmatrix} \tag{23}$$

, from which one may obtain the characteristic polynomial

$$P(\lambda)=\lambda^4+a_1\lambda^3+a_2\lambda^2+a_3\lambda+a_4 \tag{24}$$

, where

$$\begin{aligned} a_1 &= \frac{7}{2}f_4, \\ a_2 &= \frac{1}{2}\left(f_4^2+f_1+6f_2+2f_3-2\eta\right), \\ a_3 &= \frac{1}{2}\left(f_1+f_2+5f_3\right)f_4, \\ a_4 &= \frac{1}{2}f_1\left(f_2+f_3\right)+f_3\left(2f_2-\eta\right). \end{aligned} \tag{25}$$

It can be shown that at the critical point defined by

$$f_1=\frac{1}{2},\quad f_2=\frac{5}{8},\quad f_3=\frac{1}{8},\quad f_4=\frac{1}{2},\quad \eta_c=\frac{3}{2} \tag{26}$$

The polynomial $P(\lambda)$ has a pair of purely imaginary and two distinct negative eigenvalues

$$\lambda_{1,2}=\pm\frac{1}{2}i,\quad \lambda_3=-\frac{1}{2},\quad \lambda_4=-\frac{5}{4} \tag{27}$$

Shifting the parameter η by the parameter exchange of

$$\mu=\eta-\eta_c=\eta-\frac{3}{2} \tag{28}$$

, and transforming the Jacobian matrix A into its modal canonical form [22-23] using the canonical matrix B, i.e. $z=Bx$,

$$B=\begin{pmatrix} -\frac{1}{20} & \frac{7}{20} & -2 & -1 \\ -\frac{7}{40} & -\frac{1}{40} & 1 & \frac{5}{4} \\ 0 & \frac{1}{2} & 0 & 1 \\ -\frac{1}{4} & 0 & 0 & -\frac{5}{4} \end{pmatrix} \tag{29}$$

One may transform the system (22) into the system

$$\dot{x}=Jx+g(x) \tag{30}$$

, where the Jacobian canonical matrix J at the origin $x_i=0$ and at the critical point $\mu_c=0$ is given by

$$J=B^{-1}AB=\begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{5}{4} \end{pmatrix} \tag{31}$$

, and the non-linear part $g(x)$ is listed in the Appendix.

Reduction to Centre Manifold

Substituting the matrix J from the equation (31) and the set of function $g_{i,j}(x)$ from the Appendix into the equation (30) and also changing the time scale, $t=\frac{1}{2}t'$, one may obtain

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix}=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\frac{5}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} +2\begin{bmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,20} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,20} \\ g_{3,1} & g_{3,2} & \cdots & g_{3,20} \\ g_{4,1} & g_{4,2} & \cdots & g_{4,20} \end{bmatrix} X \tag{32}$$

, where the vector X is defined as

$$X=[x_1^3,\ x_2^3,\ x_3^3,\ x_4^3,\ x_1^2x_2,\ x_1^2x_3,\ x_1^2x_4, x_2^2x_1,\ x_2^2x_3,\ x_2^2x_4,\ x_3^2x_1,\ x_3^2x_2,\ x_3^2x_4, x_4^2x_1,\ x_4^2x_2,\ x_4^2x_3,\ x_1x_2x_3,\ x_1x_2x_4, x_1x_3x_4,\ x_2x_3x_4]^{\mathrm{T}} \tag{33}$$

, and the coefficients of the function $g(x)$ denoted by $g_{i,j}$ are provided at the Appendix. The above system can be split into the two systems of equations of the form

$$\begin{aligned} \dot{u} &= J_1u+f_1(u,v)=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}u+G_u \\ \dot{v} &= J_2v+f_2(u,v)=\begin{bmatrix} -1 & 0 \\ 0 & -\frac{5}{2} \end{bmatrix}v+G_v \end{aligned} \tag{34}$$

, where

$$\begin{aligned} G_u &= 2\begin{bmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,20} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,20} \end{bmatrix}X \\ G_v &= 2\begin{bmatrix} g_{3,1} & g_{3,2} & \cdots & g_{3,20} \\ g_{4,1} & g_{4,2} & \cdots & g_{4,20} \end{bmatrix}X \end{aligned} \tag{35}$$

Here, $u=(x_1,x_2)$ and $v=(x_3,x_4)$ are the split co-ordinates and $u=(x_1,x_2)$ is assumed to be the corresponding critical mode.

Since a solution near the origin $x=(0,0,0,0)$ is sought, if an approximate function $v=h(u)$ can be found near the origin such that the following relationship valid

$$\begin{aligned} \dot{v} &= D_uh(u)\dot{u}=D_uh(u)[J_1u+f_1(u,h(u))] \\ &= J_2h(u)+f_2(u,h(u)) \end{aligned} \tag{36}$$

, then the first equation in (34) is only needed for the critical mode. In the above relation, the matrix $D_u h(u)$ is the Jacobian matrix of $h(u)$.

Normal Form Coefficients

The function $h(u)$ can be approximated by a third order function of the form

$$v=\begin{bmatrix}x_3\\x_4\end{bmatrix}=h(u)=\begin{bmatrix}h_1\\h_2\end{bmatrix}=\begin{bmatrix}a_{10}&a_{01}\\b_{10}&b_{01}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}a_{20}&a_{11}&a_{02}\\b_{20}&b_{11}&b_{02}\end{bmatrix}\begin{bmatrix}x_1^2\\x_1x_2\\x_2^2\end{bmatrix}+\begin{bmatrix}a_{30}&a_{21}&a_{12}&a_{03}\\b_{30}&b_{21}&b_{12}&a_{03}\end{bmatrix}\begin{bmatrix}x_1^3\\x_1^2x_2\\x_1x_2^2\\x_2^3\end{bmatrix}\tag{37}$$

where a_{ij} and b_{ij} are unknown constants. Moreover, the Jacobian matrix, $D_u h(u)$, can be determined as follows

$$D_u h(u)=\begin{bmatrix}D_{11}&D_{12}\\D_{21}&D_{22}\end{bmatrix}=\begin{bmatrix}\frac{\partial h_1}{\partial x_1}&\frac{\partial h_1}{\partial x_2}\\\frac{\partial h_2}{\partial x_1}&\frac{\partial h_2}{\partial x_2}\end{bmatrix}\tag{38}$$

where

$$\begin{aligned}D_{11}&=a_{10}+2a_{20}x_1+a_{11}x_2+3a_{30}x_1^2+2a_{21}x_1x_2+a_{12}x_2^2\\D_{12}&=a_{01}+2a_{02}x_2+a_{11}x_1+3a_{03}x_2^2+2a_{12}x_1x_2+a_{21}x_1^2\\D_{21}&=b_{10}+2b_{20}x_1+b_{11}x_2+3b_{30}x_1^2+2b_{21}x_1x_2+b_{12}x_2^2\\D_{12}&=b_{01}+2b_{02}x_2+b_{11}x_1+3b_{03}x_2^2+2b_{12}x_1x_2+b_{21}x_1^2\end{aligned}\tag{39}$$

Substituting the equation (37-39) into (36), the following equation is obtained.

$$\begin{bmatrix}D_{11}&D_{12}\\D_{21}&D_{22}\end{bmatrix}\left\{\begin{pmatrix}x_2\\-x_1\end{pmatrix}+\begin{pmatrix}G_u(1)\\G_u(2)\end{pmatrix}\right\}=\begin{pmatrix}-h_1\\-\frac{5}{2}h_2\end{pmatrix}+\begin{pmatrix}G_v(1)\\G_v(2)\end{pmatrix}\tag{40}$$

, where the functions, G_u and G_u , are third order functions. Implementing (37) and (39) into (40) and ignoring all terms of order $O(|x^4|)$ and higher, a set of very long algebraic equations for the unknown coefficients in (37) are obtained by equating the coefficients of all similar power terms in left and right hand side of (40). This set of algebraic equations was solved using an algebraic operating code in Mathematica [24]. Follow is the results obtained for unknown coefficients in (37).

$$\begin{bmatrix}a_{10}&a_{01}\\b_{10}&b_{01}\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\begin{bmatrix}a_{20}&a_{11}&a_{02}\\b_{20}&b_{11}&b_{02}\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix}a_{30}&a_{21}&a_{12}&a_{03}\\b_{30}&b_{21}&b_{12}&b_{03}\end{bmatrix}=\begin{bmatrix}\frac{-1079}{512000}&\frac{11}{128000}&\frac{297}{512000}&\frac{593}{768000}\\\frac{9894137}{4924896000}&\frac{3288671}{1641632000}&\frac{764393}{1641632000}&\frac{1880373}{1641632000}\end{bmatrix}\tag{41}$$

The reduced system is therefore (up to third order terms) becomes

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\varepsilon\begin{bmatrix}\left(-\frac{83069}{11136000}x_1^3-\frac{88487}{3712000}x_1^2x_2+\frac{231}{128000}x_1x_2^2-\frac{931423}{11136000}x_2^3\right)\\\left(-\frac{16811}{5568000}x_1^3-\frac{18753}{1856000}x_1^2x_2-\frac{111}{64000}x_1x_2^2-\frac{6179}{1856000}x_2^3\right)\end{bmatrix}\tag{42}$$

This system is shifted from the origin based on the system parameter μ and then analyzed for limit cycle oscillations.

Results and Discussion

The reduced system (42) is shifted from the origin based on the system parameter μ and then analysed for predicting limit cycle oscillations according the discussion above and the developed Mathematica programme. The resulted centre manifold equation is given below.

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}\frac{8}{145}\mu x_1+\frac{24}{145}\mu x_2\\\frac{9}{145}\mu x_1+\frac{27}{145}\mu x_2\end{bmatrix}+\varepsilon\begin{bmatrix}\left(-\frac{83069}{11136000}x_1^3-\frac{88487}{3712000}x_1^2x_2+\frac{231}{128000}x_1x_2^2-\frac{931423}{11136000}x_2^3\right)\\\left(-\frac{16811}{5568000}x_1^3-\frac{18753}{1856000}x_1^2x_2-\frac{111}{64000}x_1x_2^2-\frac{6179}{1856000}x_2^3\right)\end{bmatrix}\tag{43}$$

Taking μ=0.1 and ε=1 and by solving up to three order normal forms, the following normal form solution is obtained

$$\begin{bmatrix}\dot{r}\\\dot{\theta}\end{bmatrix}=\begin{bmatrix}0.012069\,r-0.00523608\,r^3\\-0.00500178+0.0314786\,r^2\end{bmatrix}\tag{44}$$

The steady state solution of $(r,\theta)=(1.51821, 0.0676t)$ is obtained from the above equation. Implementing the steady state solution and reversing the co-ordinate transformations, the solution to the original system (physical system) for the limit cycles is obtained as follows

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} -0.0670007\,\mathrm{Cos}[0.432445\,t] \\ -0.00424099\,\mathrm{Cos}[1.29733\,t] \\ -0.568605\,\mathrm{Sin}[0.432445\,t] \\ -0.00449492\,\mathrm{Sin}[1.29733\,t] \end{pmatrix} \\ \begin{pmatrix} -0.247071\,\mathrm{Cos}[0.432445\,t] \\ -0.0067542\,\mathrm{Cos}[1.29733\,t] \\ +0.0424101\,\mathrm{Sin}[0.432445\,t] \\ +0.00167113\,\mathrm{Sin}[1.29733\,t] \end{pmatrix} \\ \begin{pmatrix} 0.00502747\,\mathrm{Cos}[0.432445\,t] \\ -0.00323571\,\mathrm{Cos}[1.29733\,t] \\ -0.81301\,\mathrm{Sin}[0.432445\,t] \\ -0.00696134\,\mathrm{Sin}[1.29733\,t] \end{pmatrix} \\ \begin{pmatrix} -0.3526\,\mathrm{Cos}[0.432445\,t] \\ -0.00987998\,\mathrm{Cos}[1.29733\,t] \\ +0.00251373\,\mathrm{Sin}[0.432445\,t] \\ +0.00189009\,\mathrm{Sin}[1.29733\,t] \end{pmatrix} \end{bmatrix} \tag{45}$$

The results are compared with the Runge-Kutta numerical solution at two set of initial conditions. The first initial condition is set a point outside the LCO as

IC1: $(x_1, x_2, x_3, x_4) = (-1.70,0,0,0)$
or $(z_1, z_2, z_3, z_4) = (0.085,0.2975,0.0,0.425)$

, and the second set of initial condition correspond to a point inside LCO as

IC2: $(x_1, x_2, x_3, x_4) = (-0.5,0,0,0)$
or $(z_1, z_2, z_3, z_4) = (0.025,0.0875,0.0,0.125)$

The numerical results are compared with the analytical solutions obtained from normal forms in Figures 2 and 3, for IC1 and IC2 conditions, respectively.

Conclusion

A study has been carried out for calculating limit cycle oscillations for a two-degree-of-freedom nonlinear double pendulum system. The order of system is reduced by a centre manifold approach corresponding to the critical mode of the system. Normal forms were successfully obtained by a period averaging method for the reduced system. Normal forms and limit cycle oscillations were obtained for the reduced system using an algebraic programme in Mathematica. The numerical simulations for the full system using the Runge-Kutta method were compared with LCO solutions from the analytical approach. The analytical normal form estimations are in good agreement with the numerical results. Apart from run-time speed and the required memory of machine, no noticeable difficulty has been observed for this class of problems using Mathematica. The developed symbolic code can be trusted for large systems using a high capacity computer. Further research for developing a general reduction code to deal with large dynamical systems is in order.

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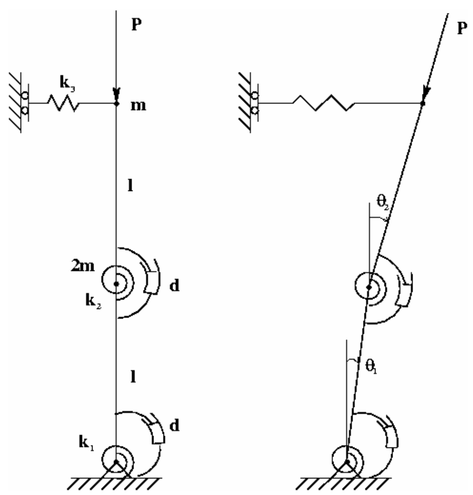

Fig. 1 - A sketch of a double pendulum system with follower force

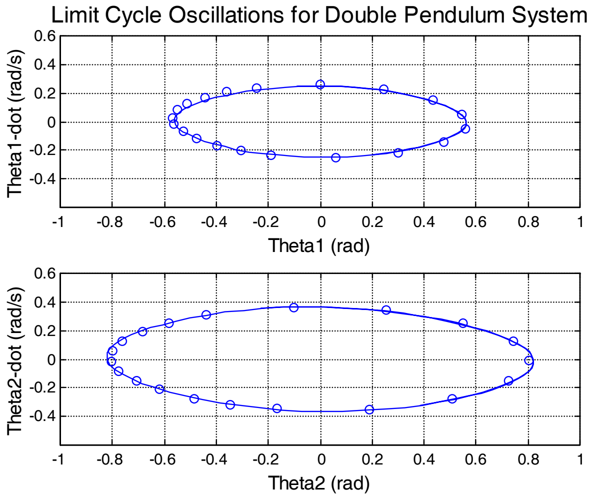

Fig. 2 - The comparison of the analytical NFT, symbol, and the numerical Runge-Kutta, solid line, limit cycle solutions (IC1).

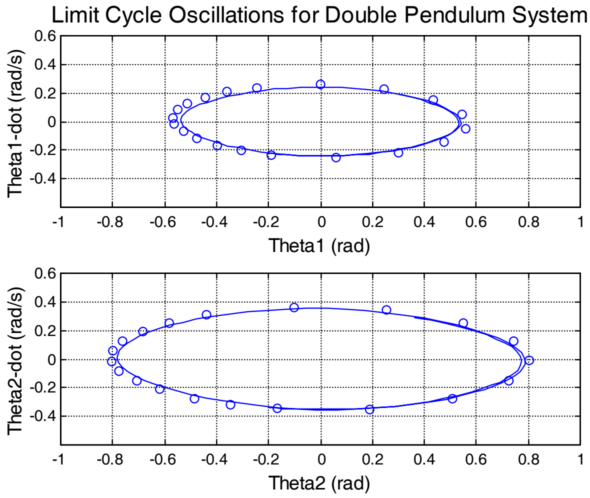

Fig. 3 - The comparison of the analytical NFT, symbol, and the numerical Runge-Kutta, solid line, limit cycle solution (IC2).

Appendix

$$g_1 = \begin{bmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,20} \end{bmatrix} X =$$
$$= -\frac{83069}{11136000}x_1^3 - \frac{931423}{11136000}x_2^3 + \frac{631}{174}x_3^3 + \frac{1318}{87}x_4^3$$
$$-\frac{88487}{3712000}x_1^2 x_2 - \frac{23169}{92800}x_1^2 x_3 - \frac{102613}{3712000}x_1^2 x_4$$
$$+\frac{231}{128000}x_1 x_2^2 + \frac{17799}{92800}x_2^2 x_3 - \frac{125837}{3712000}x_2^2 x_4$$
$$+\frac{4731}{2320}x_1 x_3^2 - \frac{1087}{2320}x_2 x_3^2 + \frac{4587}{232}x_3^2 x_4 + \frac{1185}{928}x_1 x_4^2$$
$$+\frac{305}{928}x_2 x_4^2 + \frac{3725}{116}x_3 x_4^2 + \frac{3213}{46400}x_1 x_2 x_3$$
$$+\frac{3041}{185600}x_1 x_2 x_4 + \frac{14987}{46400}x_1 x_3 x_4 + \frac{3841}{4640}x_2 x_3 x_4;$$

$$g_2 = \begin{bmatrix} g_{2,1} & g_{2,2} & \cdots & g_{2,20} \end{bmatrix} X =$$
$$= -\frac{16811}{5568000}x_1^3 - \frac{6179}{1856000}x_2^3 - \frac{71}{87}x_3^3 - \frac{221}{87}x_4^3$$
$$-\frac{18753}{1856000}x_1^2 x_2 - \frac{4831}{46400}x_1^2 x_3 - \frac{22077}{1856000}x_1^2 x_4$$
$$-\frac{111}{64000}x_1 x_2^2 - \frac{999}{46400}x_2^2 x_3 - \frac{13173}{185600}x_2^2 x_4$$
$$+\frac{749}{1160}x_1 x_3^2 - \frac{1273}{1160}x_2 x_3^2 - \frac{637}{116}x_3^2 x_4 + \frac{5}{464}x_1 x_4^2$$
$$-\frac{635}{464}x_2 x_4^2 - \frac{415}{58}x_3 x_4^2 - \frac{413}{23200}x_1 x_2 x_3$$
$$-\frac{5911}{92800}x_1 x_2 x_4 + \frac{1443}{2320}x_1 x_3 x_4 - \frac{5751}{2320}x_2 x_3 x_4;$$

$$g_3 = \begin{bmatrix} g_{3,1} & g_{3,2} & \cdots & g_{3,20} \end{bmatrix} X =$$
$$= -\frac{1123}{1024000}x_1^3 + \frac{2077}{3072000}x_2^3 - \frac{7}{16}x_3^3 - \frac{35}{24}x_4^3$$
$$-\frac{3787}{1024000}x_1^2 x_2 - \frac{973}{25600}x_1^2 x_3 - \frac{4469}{102400}x_1^2 x_4$$
$$-\frac{801}{1024000}x_1 x_2^2 - \frac{357}{25600}x_2^2 x_3 - \frac{2221}{102400}x_2^2 x_4$$
$$+\frac{143}{640}x_1 x_3^2 - \frac{291}{640}x_2 x_3^2 - \frac{181}{64}x_3^2 x_4 - \frac{7}{256}x_1 x_4^2$$
$$-\frac{151}{256}x_2 x_4^2 - \frac{123}{32}x_3 x_4^2 - \frac{119}{12800}x_1 x_2 x_3$$
$$-\frac{1407}{51200}x_1 x_2 x_4 + \frac{47}{256}x_1 x_3 x_4 - \frac{275}{256}x_2 x_3 x_4;$$

$$g_4 = \begin{bmatrix} g_{4,1} & g_{4,2} & \cdots & g_{4,20} \end{bmatrix} X =$$
$$= \frac{16811}{11136000}x_1^3 + \frac{6179}{3712000}x_2^3 + \frac{71}{174}x_3^3 + \frac{221}{174}x_4^3$$
$$+\frac{18753}{3712000}x_1^2 x_2 - \frac{4831}{92800}x_1^2 x_3 + \frac{22077}{371200}x_1^2 x_4$$
$$+\frac{111}{128000}x_1 x_2^2 + \frac{999}{92800}x_2^2 x_3 + \frac{13173}{371200}x_2^2 x_4$$
$$-\frac{749}{2320}x_1 x_3^2 + \frac{1273}{2320}x_2 x_3^2 + \frac{637}{232}x_3^2 x_4 - \frac{5}{928}x_1 x_4^2$$
$$+\frac{635}{928}x_2 x_4^2 + \frac{415}{116}x_3 x_4^2 + \frac{413}{46400}x_1 x_2 x_3$$
$$+\frac{5911}{185600}x_1 x_2 x_4 - \frac{1443}{4640}x_1 x_3 x_4 + \frac{5751}{4640}x_2 x_3 x_4.$$