LIMIT CYCLE OSCILLATION PREDICTION FOR NON-LINEAR AEROELASTIC SYSTEMS

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Abstract

This paper describes part of an investigation into the prediction and characterisation of Limit Cycle Oscillations occurring in non-linear aeroelastic systems. Through the use of a modified version of Normal Form Theory, it is shown how it is possible to predict the Limit Cycle Oscillations and characterise their stability. The approach is analytical and does away with the need for numerical simulation of the system. The methodology is demonstrated upon a simple two degrees-of-freedom aeroelastic wing model with cubic stiffness. A good agreement was obtained between the analytical prediction and numerical simulations.

Introduction

Classical theory assumes linear aerodynamics interacting with a linear structure for modelling the aeroelastic behaviour. However, the influence of non-linearities on modern aircraft is becoming of increasing importance (1). These non-linearities can be structural (free-play, backlash, cubic stiffness), aerodynamic (non-linear damping, moving shocks) or control (time delays, non-linear control laws) based and can result in behaviour such as Limit Cycle Oscillations (LCO) that cannot occur in a linear system (2,3). With the increasing use of increasingly sophisticated technology (e.g. stealth) along with construction techniques that reduce the amount of inherent structural damping, the use of linear analysis techniques for aeroelastic analysis is becoming less feasible. LCO has been an important topic for a number of years, see for instance the last two proceedings of the International Forum on Aeroelasticity and Structural Dynamics (1997, 1999) (4-7). It is of particular interest to determine whether an aircraft will experience Limit Cycle Oscillations and/or Flutter, whereabouts in the flight envelope these phenomena will occur, and the precise nature of the instabilities. Current computational techniques need extremely large amounts of coupled aerodynamic/structural computation to predict non-linear aeroelastic behaviour. The ability to accurately characterise LCO and to predict the stability boundaries is very important. Although not desirable, LCO is essentially a fatigue problem, whereas flutter is usually catastrophic and must be avoided at all costs. An accurate LCO/flutter prediction capability would reduce significantly the amount of flights required in any flight clearance test programme with current costs being estimated at around $70k per test flight.

There has been much work in recent years (e.g.2-7) devoted towards the characterisation of non-linear aeroelastic behaviour, including LCO. This work has primarily consisted of simulating the response of the aeroelastic system through numerical integration, although there are a few known instances of experimental verification (8). Improved unsteady CFD modelling allied to the coupling of the aerodynamic and structural grids (9-11) has made significant headway towards solving the problem, particularly in the Transonic region. However, there are still major problems inherent in such an approach due to the enormous computational resources required for even the simplest cases. The harmonic balance method (12) is a relatively simple approach that begins to address the problem, however, due to the assumption made in modelling the non-linearity, the method does not produce accurate estimates of the LCO behaviour.

A number of mathematical techniques exist in the non-linear dynamics community that enables the stability boundaries of a known non-linear system to be defined and also the possible instabilities to be characterised. Of particular relevance is Normal Form theory (13), although this method can only be applied to systems with non-linearities that are continuous.

This paper describes an approach to determine the shape and characteristics of limit cycle oscillations in non-linear aeroelastic systems without the need for extensive computational simulations. The methodology that has been developed is not seen as replacing the extensive CFD modelling mentioned above, but as a guide to which parts of the flight envelope that should be investigated using sophisticated CFD methods. This work is part of a research programme aimed at developing a complete modelling and predictive capability for non-linear aircraft.

The methodology of the proposed approach is described using a simple binary system with a structural non-linearity. The procedure involves transforming the system equations into modal canonical form, reducing these equations into Normal Form, and then predicting the limit cycle oscillations. The Normal Form approach has to be reformulated to be able to analyse aeroelastic systems. The predicted results are compared with those from numerical simulation.

Governing Equations for a Two DOF aeroelastic system

Consider a simple wing model (14) shown in Figure 1, representing a wing with rigid chordwise sections with semi-span *s* and chord, *c*. The wing model is fixed at its root with bending γ and torsion θ degrees of freedom. In general, the equations of motion for the wing may be found using Lagrange equations such that

$$\begin{bmatrix} \frac{ms^5c}{5} & 0 \\ 0 & \frac{ms^3c^3}{36} \end{bmatrix} \begin{pmatrix} \ddot{\gamma} \\ \ddot{\theta} \end{pmatrix} + \begin{bmatrix} -\frac{\rho VcL_{\dot{z}}s^5}{10} & -\frac{\rho VL_{\dot{\theta}}c^2s^4}{8} \\ -\frac{\rho Vc^2M_{\dot{z}}s^4}{8} & -\frac{\rho Vc^3L_{\dot{z}}s^3}{72}-\frac{\rho VM_{\dot{\theta}}s^3c^3}{6} \end{bmatrix} \begin{pmatrix} \dot{\gamma} \\ \dot{\theta} \end{pmatrix} + \begin{bmatrix} 4EIs-\frac{\rho V^2L_zs^5}{10} & -\frac{\rho V^2L_{\theta}cs^4}{8} \\ -\frac{\rho V^2cM_zs^4}{8} & GJs-\frac{\rho V^2c^2L_zs^3}{72}-\frac{\rho V^2M_{\theta}s^3c^2}{6} \end{bmatrix} \begin{pmatrix} \gamma \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1)$$

where ρ is the air density, *V* is the air speed, and *m* is the mass per unit length of the wing. The oscillatory aerodynamics forces can be expressed (15) as functions of the reduced frequency $\nu = \omega c / V$ such that

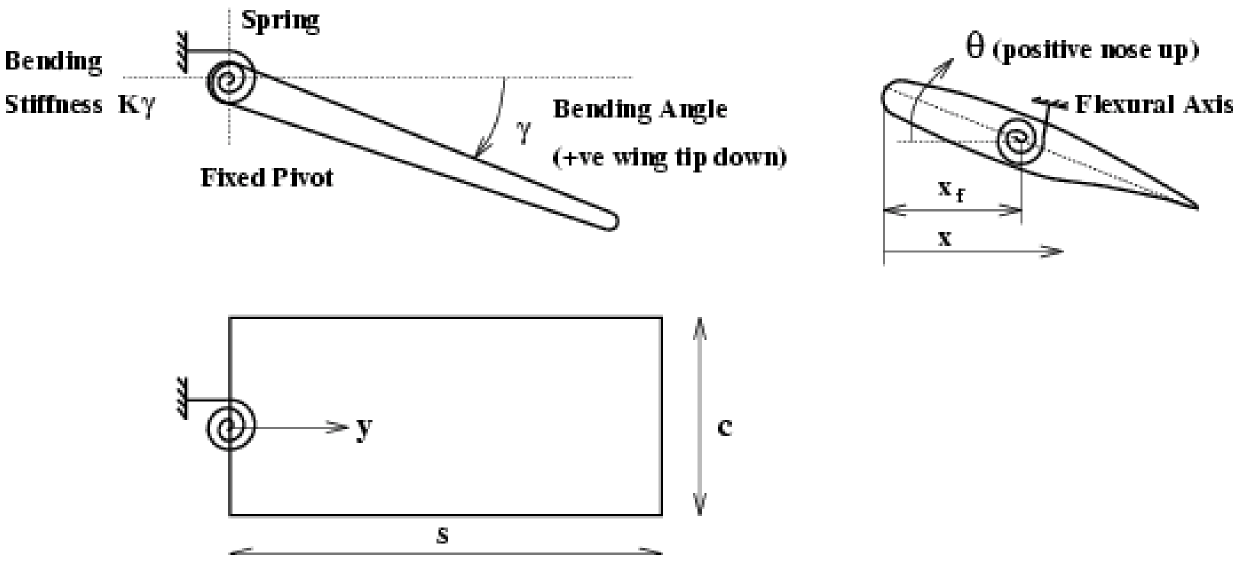

Figure 1. Schematic of the rectangular wing model

$$
\begin{aligned}
L_z &= 2\pi\left[-\frac{\nu^2}{4} - \nu G(\nu/2)\right] \\
L_{\dot z} &= 2\pi F(\nu/2) \\
L_\theta &= 2\pi\left[-\frac{\nu^2}{4}\left(e-\frac{1}{4}\right) + F(\nu/2) - \nu G(\nu/2)\left(\frac{1}{2}-e\right)\right] \\
L_{\dot\theta} &= 2\pi\left[\frac{1}{4} + F(\nu/2)\left(\frac{1}{2}-e\right) + \frac{G(\nu/2)}{\nu}\right]
\end{aligned} \qquad (2)
$$

in which e is the distance of flexural axis from the aerodynamic centre divided by the chord length and $F(\nu)$ and $G(\nu)$ are the components of Theodorsen's function $C(\nu)=F(\nu)+iG(\nu)$. In the same way the moment derivatives are given as

$$
\begin{aligned}
M_z &= 2\pi\left[-\frac{\nu^2}{4}\left(e-\frac{1}{4}\right) - e\nu G(\nu/2)\right] \\
M_{\dot z} &= 2\pi\left[eF(\nu/2)\right] \\
M_\theta &= 2\pi\left[\frac{\nu^2}{4}\left(\left(e-\frac{1}{4}\right)^2 + \frac{1}{32}\right) + eF(\nu/2) - e\nu G(\nu/2)\left(\frac{1}{2}-e\right)\right] \\
M_{\dot\theta} &= 2\pi\left[-\frac{1}{4}\left(\frac{1}{2}-e\right) + eF(\nu/2)\left(\frac{1}{2}-e\right) + e\frac{G(\nu/2)}{\nu}\right]
\end{aligned} \qquad (3)
$$

As the aim of this paper is to illustrating the basic concepts of the approach, only simplistic aerodynamics will be used (14). Quasi-steady strip theory is used but with the $M_{\dot\theta}$ term included to allow for unsteady effects.

The general two DOF aeroelastic system including a non-linear function $F(q)$ can be expressed (15) as

$$
\mathbf{A}\,\ddot q + (\rho V\mathbf{B} + \mathbf{D})\,\dot q + (\rho V^2\mathbf{C} + \mathbf{E})q + F(q) = 0 \qquad (4)
$$

where $q=(q_1, q_2)=(\gamma, \theta)$, $F(q)$ is any continuous non-linear function of q. Matrices A, B, D, C and E are the mass, the aerodynamic damping, the structural damping, the aerodynamic stiffness and the structural stiffness matrices, respectively, defined as

$$
\mathbf{A} = \begin{bmatrix} I_\gamma & I_{\gamma\theta} \\ I_{\gamma\theta} & I_\theta \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \frac{cs^2a}{6} & 0 \\ -\frac{c^2s^2ea}{4} & -\frac{c^3s}{2}M_{\dot\theta} \end{bmatrix} \qquad (5)
$$

$$
\mathbf{C} = \begin{bmatrix} 0 & \frac{cs^2a}{4} \\ 0 & -\frac{c^2sea}{2} \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} k_\gamma & 0 \\ 0 & k_\theta \end{bmatrix}
$$

where I_γ, I_θ, and $I_{\gamma\theta}$ are the moments of inertia in bending, in pitch and their product, respectively. k_γ, k_θ, are rotational stiffness in bending and torsion, respectively. The parameters e and a are the non-dimensional distance of flexural axis from aerodynamic centre, and the two-dimensional sectional lift curve slope, respectively. The structural damping has been ignored here, however, it can be included without reducing the generality of the following analysis.

Liapunov-Schmidt Reduction Method

The normal form method used here only considers a single degree of freedom system. Consequently, for large multi-degree of freedom systems, some approach must be found to reduce the order of the equations whilst retaining the key non-linear characteristics. In this work, an approach is introduced based upon the Liapunov-Schmidt reduction method, which reduces a multi degree of freedom dynamical system into a single degree of freedom system corresponding to the critical mode. Thus, higher order normal forms can be obtained based on the reduced system.

The Hancock system of equations 4 can be determined at its critical mode and speed, taken here as the linear flutter condition. Transformation into modal canonical form gives

$$
\dot x = Jx + g(x) \qquad (6)
$$

where J is the Jacobian canonical matrix at the origin corresponding to the critical point and $g(x)$ is the non-linear part of the system. The above system can be split into two systems of equations of the form

$$\dot{u} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = J_1 u + f_1(u,v) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} u + f_1(u,v)$$
$$\dot{v} = \begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = J_2 v + f_2(u,v) = \begin{bmatrix} -a & b \\ b & -a \end{bmatrix} v + f_2(u,v) \quad (7)$$

where the first equation in 7 corresponds to the critical mode. Since a solution near the origin *x=(0,0,0,0)* is sought, if an approximate function *v = h (u)* can be found near the origin such that

$$\dot{v} = D_u h(u)\dot{u} = D_u h(u)[J_1 u + f_1(u, h(u))]$$
$$= J_2 h(u) + f_2(u, h(u)) \quad (8)$$

is valid, then only the first equation in 7 will be needed for the non-linear analysis. In the above relation, the matrix $D_u h(u)$ is the Jacobian matrix of *h(u)*. The function *h(u)* can be approximated by a third order function of the form

$$v = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = h(u) = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$
$$= \begin{bmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (9)$$
$$+ \begin{bmatrix} a_{20} & a_{11} & a_{02} \\ b_{20} & b_{11} & b_{02} \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}$$
$$+ \begin{bmatrix} a_{30} & a_{21} & a_{12} & a_{03} \\ b_{30} & b_{21} & b_{12} & a_{03} \end{bmatrix} \begin{bmatrix} x_1^3 \\ x_1^2 x_2 \\ x_1 x_2^2 \\ x_2^3 \end{bmatrix} + \cdots$$

where a_{ij} and b_{ij} are unknown constants. Substituting 9 into 8, then equating similar terms in the right and left hand sides of 8 whilst ignoring all terms of order $O(|x^4|)$ and higher, gives a set of algebraic equations for the unknown coefficients in 9. These were solved using the Mathematica algebraic package.

Normal Form Theory

Normal Form Theory (NFT) comprises of a non-linear co-ordinate transformation to obtain a simple analytical expression for the transformed equations such that the qualitative behaviour of the system is evaluated without the need for solving the system of equations. In this section, the classical NFT of Poincare (16) and Brikhoff (17) is briefly discussed for a *n* degrees of freedom non-linear system.

Consider the non-linear ordinary differential equations in modal canonical form as follows

$$\dot{x} = Jx + f(x) = Jx + f_2(x) + \cdots + f_r(x) + O\left(|x|^{r+1}\right), \; x \in R^n \quad (10)$$

where *J* is the Jordan canonical form and $f_k(x) \in H_n^k$ is the k^{th} order homogeneous polynomial in *x*.

In order to simplify the non-linear terms in equation 10 into their normal forms, a near identity non-linear co-ordinate transformation of the form

$$x = y + h_k(y), \; h_k(y) \in H_n^k, \; 2 \le k \le r \quad (11)$$

is introduced, where $h_k(y)$ is the k^{th} order function of *y*. Substituting equation 11 into (10) and truncating up to the k^{th} order gives

$$\dot{y} = Jy + f_2(y) + \cdots + f_{k-1}(y) + f_k(y) + ad_J^k h_k(y) + O\left(|y|^{k+1}\right) \quad (12)$$

where the function $ad_J^k h_k(y)$ denotes an adjacent operator equivalent to the function of the Lie Bracket (16,18,19). The normal form equations 12 are finally written in the polar co-ordinates such that

$$\begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \Omega_1(r) \\ \Omega_2(r) \end{bmatrix} \quad (13)$$

which can be evaluated for the stability and characteristics of any number of limit cycles.

Period Averaging Method

It is difficult to find the coefficients of normal forms using the matrix based NFT method discussed above. As an alternative, the averaging method (20,21) can be used which is equivalent to the NFT method. The problem of calculating higher order averaging equations is equivalent to the calculation of the higher order normal form coefficients. Since the averaging method is applied to non-autonomous systems, a co-ordinate transformation is chosen such that an autonomous system is obtained through a time integrating procedure.

In this approach, the non-linear ordinary differential equation

$$\dot{x} = Jx + \varepsilon f(x,\varepsilon), \quad x \in R^n \tag{14}$$

is transformed to

$$\dot{y} = \varepsilon e^{-tJ} f\left(e^{tJ} y, \varepsilon\right) = \varepsilon g(y,t,\varepsilon) \tag{15}$$

by using the transformation

$$x = e^{tJ} y \text{ and } \dot{x} = Je^{tJ} y + e^{tJ} \dot{y} \tag{16}$$

Note that the equation 15 is explicitly dependent on time while the original equation 14 is not. The period averaged normal forms of equation 15 can be constructed using the variable exchange

$$y = \zeta + \sum_{l=1}^{m} \varepsilon^l h_l(\zeta, t) \tag{17}$$

where $h_k(\zeta,t)$ is a geometrical transformations (22). Thus, the normal forms up to order *m* can be obtained as

$$\dot{\zeta} = \sum_{k=1}^{m} \varepsilon^k f_k^0(\zeta) + O\left(\varepsilon^{m+1}\right) \tag{18}$$

where $f_k^0(\zeta)$ is an integral function (22). The equation 18 can be written in the polar co-ordinate system similar to the equation 13.

Limit-cycle prediction

The algebraic code developed by Leung and Zhang (22) has been extended to solve aeroelastic problems. Considering the aeroelastic problems, it is found that the equation 10 is not sufficient for obtaining normal forms and therefore a modified version of the NFT method is constructed using the expression

$$\dot{x} = Jx + f(x) = Jx + f_1(x) + f_2(x) + \cdots + f_r(x) + O\left(|x|^{r+1}\right), \; x \in R^n \tag{19}$$

where $f_1(x) = J_S x$ is the shift of the linear part of system from the origin to the critical condition and is included into the non-linear part.

Numerical example with cubic stiffness

Adding a cubic structural stiffness in torsion, the binary system can be rewritten as

$$\mathbf{A}\ddot{q} + (\rho V\mathbf{B} + \mathbf{D})\dot{q} + \left(\rho V^2\mathbf{C} + \mathbf{E}\right)q + \mathbf{EN}q^3 = 0 \tag{20}$$

with

$$\mathbf{EN} = \begin{bmatrix} 0 & 0 \\ 0 & k_\theta \end{bmatrix} \tag{21}$$

where the following numerical values for the wing model are used

$$\begin{gathered} \mathrm{s} = 10\,\mathrm{m},\ \mathrm{c} = 3\,\mathrm{m},\ \mathrm{x}_{\mathrm{cm}} = 0.6\mathrm{c\ m},\ \mathrm{x}_{\mathrm{f}} = 0.5\mathrm{c\ m}, \\ \mathrm{m} = 200\,\mathrm{Kg},\ \mathrm{a} = 2\pi,\ \rho = 1.225\,\mathrm{Kg/m}^3 \end{gathered} \tag{22}$$

where here *m* is now the mass of the wing and (x_{cm},y_{cm}) is co-ordinate components of wing centre of mass. x_f is the distance of flexural axis from wing leading edge. The moments of inertia and stiffness coefficients are determined as

$$\begin{aligned} I_\gamma &= \frac{ms^2}{3} = 6.667\times10^3 \quad (\mathrm{Kg.m}^2) \\ I_\theta &= mc^2 x^{\frac{2}{3}}{}_{cm} + m\left(x_{cm} - x_f\right)^2 c^2 \\ &= 2.8255\times10^3 \quad (\mathrm{Kg.m}^2) \\ I_{\gamma\theta} &= m\left(x_{cm} - x_f\right)0.45sc = 8.1\times10^2 \quad (\mathrm{Kg.m}^2) \\ k_\gamma &= (4\pi)^2 I_\gamma = 1.0528\times10^6 \quad (\mathrm{Kg.m}^2/s^2) \\ k_\theta &= (20\pi)^2 \frac{mc^2}{12} = 5.922\times10^5 \quad (\mathrm{Kg.m}^2/s^2) \end{aligned} \tag{23}$$

For the case of constant $M_{\dot{\theta}}$ = -0.3, typical vg plots are shown in figure 2. The linear flutter speed was found to be V_F = 20.91 m/s using a direct calculation process (23).

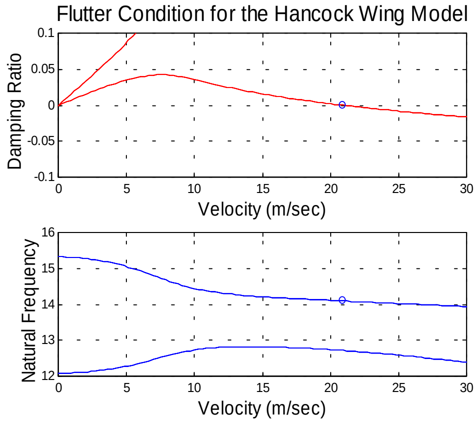

Figure 2. Damping and Frequency Variation with Airspeed

The equations of motion 20 at the flutter condition were transformed to modal canonical form and then reduced using the process described above. The unknown coefficients for this reduction obtained using the algebraic code were

```
{{a20→0.`, a02→0.`, b20→0.`, b02→0.`, a30→0.000817155575348142448`,
  a03→-0.000138034500769368229`, b30→-0.000230135787379864975`,
  b03→0.000886095492987251098`, a11→0.`, b11→0.`,
  a21→-0.000500508871650815567`, a12→0.000744361686138177436`,
  b21→0.00111499636823523196`, b12→-0.000311271023376896227`,
  b10→0.`, a01→0.`, a10→0.`, b01→0.`}}
```

The final reduced system is thus obtained as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}14.5519\varepsilon(0.007723x_1+0.06848x_2)^3\\ -2.32006\varepsilon(0.007723x_1+0.06848x_2)^3\end{bmatrix} \quad (24)$$

The above equation is shifted from the origin to the critical condition of the bifurcation parameter, in this case the airspeed. The equations are then used to calculate the LCO.

The resultant system then becomes

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \quad (25)$$
$$+\begin{bmatrix}c_1x_1+c_2x_2+14.5519\varepsilon(0.007723x_1+0.06848x_2)^3\\ c_3x_1+c_4x_2+-2.32006\varepsilon(0.007723x_1+0.06848x_2)^3\end{bmatrix}$$

where

$$\begin{bmatrix}c_1\\ c_2\\ c_3\\ c_4\end{bmatrix}=\begin{bmatrix}0.0010188-3.46439\times10^{-6}V^2\\ 0.00903291-0.000030716V^2\\ -0.00535747+0.0000182178\,V^2\\ -0.0475005+0.000161524V^2\end{bmatrix} \quad (26)$$

Here, the air speed V is assumed greater than the linear flutter speed. Applying the NFT method for up to third order normal forms, the following normal form solution is obtained

$$\begin{bmatrix}\dot{r}\\ \dot{e}\end{bmatrix}=\begin{bmatrix}\left(\frac{1}{2}(c_1+c_4)\,r-0.000083009\,r^3+9.23785\times10^{-6}c_1\,r^3+0.0000404634\,c_2\,r^3+0.0000404634\,c_3\,r^3-9.23785\times10^{-6}\,c_4\,r^3\right)\\ \left(0.125c_1^2-0.0625\,c_1^2\,c_2+0.125\,c_2^2-0.0625\,c_2^3+0.0625\,c_1^2\,c_3+0.25\,c_2c_3-0.0625\,c_2^2c_3+0.125\,c_3^2+0.0625\,c_2c_3^2+0.0625\,c_3^3+\frac{1}{2}(-c_2+c_3)-0.25\,c_1c_4+0.125\,c_1c_2c_4-0.125\,c_1c_3c_4+0.125\,c_4^2-0.0625\,c_2c_4^2+0.0625\,c_3c_4^2-0.00180817\,r^2+0.000442915\,c_1\,r^2+0.00175357\,c_2r^2+0.00175357\,c_3\,r^2-0.000442915\,c_4\,r^2\right)\end{bmatrix} \quad (27)$$

The steady state solution for the amplitude of the limit cycles in the transformed domain is obtained as

$$r=\frac{\sqrt{0.5(c_1+c_4)}}{\sqrt{0.000083009-9.23785\times10^{-6}(c_1-c_4)-0.0000404634(c_2+c_3)}} \quad (28)$$

The amplitude of limit cycle oscillations corresponding to the critical mode, i.e. the torsion mode, is calculated and plotted against the air speed V in the transformed co-ordinates in the Figure 3 and in the physical domain in Figure 4, respectively, using the algebraic code and numerical calculations.

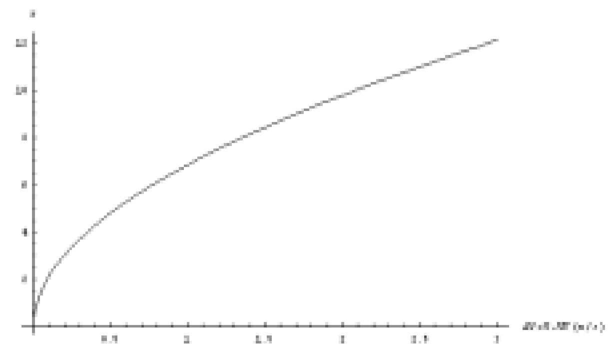

Figure 3. The amplitude of limit cycle oscillations in the transformed domain.

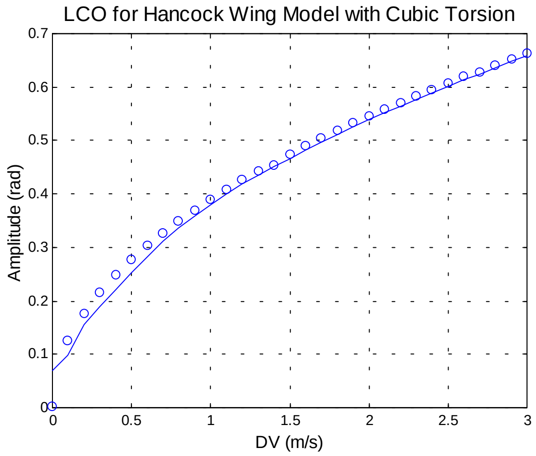

Figure 4. The amplitude of limit cycle oscillations in the physical domain using analytical (oo) and numerical (—) results.

Implementing the steady state solution and reversing all the co-ordinate transformations, the limit cycle oscillation solution for the system in physical coordinates at ΔV=3.0 is obtained as

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} \gamma \\ \theta \\ \dot{\gamma} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \left(\begin{array}{l} 0.0166662\,\mathrm{Cos}[17.9475\,t] + 0.0008498\,\mathrm{Cos}[53.84228\,t] - 0.073612 \\ \quad \mathrm{Sin}[17.9475\,t] - 0.00446217\,\mathrm{Sin}[53.84238\,t] \end{array} \right) \\ \left(\begin{array}{l} 0.7393676\,\mathrm{Cos}[17.9475\,t] + 0.019727\,\mathrm{Cos}[53.84238\,t] + 0.028504 \\ \quad \mathrm{Sin}[17.9475\,t] + 0.000067977\,\mathrm{Sin}[53.84238\,t] \end{array} \right) \\ \left(\begin{array}{l} -0.78544\,\mathrm{Cos}[17.9475\,t] - 0.01959137\,\mathrm{Cos}[53.84238\,t] - 0.280468 \\ \quad \mathrm{Sin}[17.9475\,t] - 0.015112\,\mathrm{Sin}[53.84238\,t] \end{array} \right) \\ \left(\begin{array}{l} 0.5827424\,\mathrm{Cos}[17.9475\,t] + 0.09027375\,\mathrm{Cos}[53.84238\,t] - 13.6730344 \\ \quad \mathrm{Sin}[17.9475\,t] - 0.823233\,\mathrm{Sin}[53.84238\,t] \end{array} \right) \end{bmatrix} \quad (29)$$

The form of the limit cycles was determined using both the Runge-Kutta numerical approach and the analytical method and compared in the Figures 5 and 6 using two different sets of initial conditions:

IC1: $(x_1, x_2, x_3, x_4) = (0,0,0,5.0)$ and $\Delta V = 3.0$ m/sec, and

IC2: $(x_1, x_2, x_3, x_4) = (0,0,0,20.0)$ and $\Delta V = 3.0$ m/sec.

The general shape of limit cycle is well predicated. Note that the solution has been obtained using only up to third order normal forms.

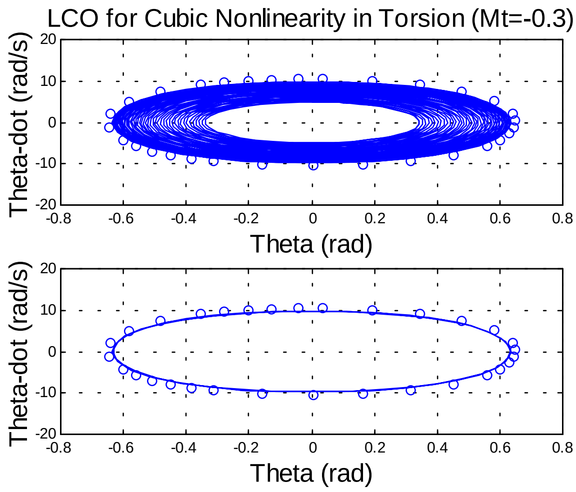

Figure 5. The comparison of the analytical NFT (ooo) and the numerical Runge-Kutta (—) limit cycle solution for the Hancock wing model (Initial Condition 1, **IC1**).

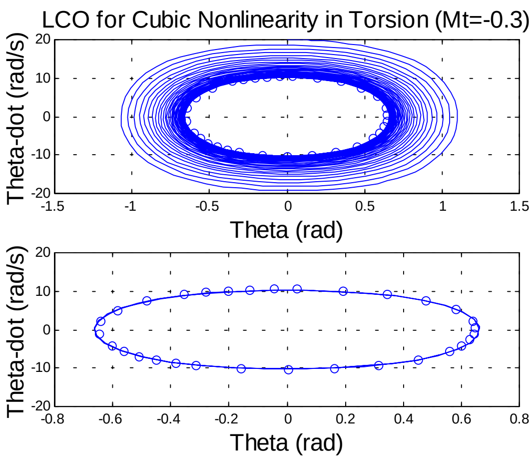

Figure 6. The comparison of the analytical NFT (ooo) and the numerical Runge-Kutta (—) limit cycle solution for the Hancock wing model (Initial Condition 2, **IC2**).

Conclusions

It has been shown how to use Normal Form Theory to predict the Limit Cycle Oscillation behaviour of aeroelastic systems with continuous non-linearities. The method was demonstrated on a simple example using a binary system with cubic stiffness; excellent agreement between the prediction and numerical simulation was achieved.

Further work is continuing to extend the approach to deal with large order systems and more realistic non-linear aerodynamic effects.

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