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AN INVESTIGATION INTO PRECONDITIONING ITERATIVE SOLVERS FOR HYDRODYNAMIC PROBLEMS*

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Abstract– Two Krylov subspace iterative methods, GMRES and QMR, were studied in conjunction with several preconditioning techniques for solving the linear system raised from the finite element discretisation of incompressible Navier-Stokes equations for hydrodynamic problems. The preconditioning methods under investigation were the incomplete factorisation methods such as ILU(0) and MILU, the Stokes preconditioner, and the Elman-Silvester block triangular preconditioner. It is observed that the GMRES solver with the Elman-Silvester preconditioner provides faster convergence than the other methods studied here.

Keywords– Krylov subspace, Navier-Stokes, preconditioned iterative methods, finite element

1. INTRODUCTION

The linear system of equations derived from the finite-element discretisation of the incompressible Navier-Stokes equations are usually large, sparse systems. The question is how to efficiently solve such systems. One well-known method for solving linear systems is the Gaussian elimination method [1], which requires storage of all n^2 entries of the matrix A as well as approximately $2n^3/3$ arithmetic operations.

The Frontal methods [2] have been designed to avoid the big storage of n^2 as well as to decrease arithmetic operations. However for large computational domains, the efficiency of the method is reduced. It is argued that the memory and the computational requirements for solving two- or three- dimensional PDE's, involving many degrees of freedom per point, may seriously challenge the most efficient direct solvers available today.

In practice, the matrices that arise from discretising PDE's are sparse, having only a few non zeros per row. For a banded matrix similar to that obtained from the finite element discretisation of hydrodynamic problems, a banded version of Gaussian elimination [1] exist, which requires storing only the approximately $2mn$ entries inside the band and performing about $2m^2n$ operations. Here, m is the band width of the matrix A. However, the algorithm cannot take advantage of any zeros inside the band which will be filled with non zeros during the process of elimination.

In contrast to the direct solvers, iterative methods can use the advantage of sparseness in matrix-vector multiplication. A general dense matrix-vector multiplication takes n^2 operations, whilst a few n operations are required for sparse matrix-vector multiplication. The storage required is only that for the non zeros of the matrix. Also, iterative methods are easier to implement efficiently on high performance computers than direct methods.

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If the system of linear equations $Ax = b$ can be solved with a moderate number of matrix-vector multiplications, then the iterative procedure may far outperform Gaussian elimination in terms of both work and storage [3]. This is why many researchers have recently focused on developing efficient iterative solvers.

2. ITERATIVE METHODS

The iterative idea for the linear system $Ax = b$ may be well understood by rewriting the linear system as

$$x = (I - A)x + b \tag{1}$$

where the iterative procedure can be performed as follows

$$x_{k+1} = (I - A)x_k + b \tag{2}$$

With the initial guess of $x_0 = 0$, it appears that *b* belongs to the space of approximate solution for *x* defined as

$$x_1 \in \text{Span}\{b\} \tag{3}$$

The process continues so that the approximation at step *k* satisfies

$$x_k \in \text{Span}\left\{ b, Ab, \cdots, A^{k-1}b \right\}, k = 1, 2, 3, \cdots \tag{4}$$

The space presented on the right hand side in (4) is called a Krylov Subspace for the matrix *A* and initial vector *b*. If it turns out that the space (4) does not contain a good approximate solution for any moderate size *k*, or if such an approximate solution cannot be computed easily, then one might consider modifying the original system using a preconditioner matrix *M*.

Since the matrix *A*, arisen from the discretisation of the hydrodynamics problems is usually non-symmetrical and indefinite, only GMRES and QMR iterative methods were considered in this work.

3. GMRES ITERATIVE METHOD

The Generalised Minimum Residual Method (GMRES) is a non-stationary method which builds the transition from x_k to x_{k+1} based on the history of the iterations. The GMRES method is an effective method for solving non-symmetrical linear systems. For a given real non-singular matrix $A \in \Re^{n\times n}$, the sequence of orthogonal vectors $(v_1, v_2, \cdots, v_m)$, constituting the orthogonal matrix $V_m = (v_1, v_2, \cdots, v_m)$, generates and is retained such that any approximate solution $x \in x_0 + K_m(A, r_0)$ can be written as

$$x = x_0 + V_m y \tag{5}$$

where $y = (y_1, y_2, \cdots, y_m)$ and

$$K_m\left(A, r_0\right) = Span\left\{ r_0, Ar_0, A^2 r_0, \cdots, A^{m-1} r_0 \right\} \tag{6}$$

Arnoldi's procedure is used for building the orthogonal basis of the Krylov subspace K_m, which is the orthogonal vectors $(v_1, v_2, \cdots, v_m)$. Defining the square upper Hessenberg matrix $H_m \in \Re^{m\times m}$ so that the following relation is satisfied

$$H_m = V_m^T A V_m \tag{7}$$

Then, v_{i+1} can be extracted using the Arnoldi algorithm for $i = 1,2,\cdots, m-1$ as follows [4]

$$v_{i+1} = \frac{Av_i - \sum_{j=1}^{i} h_{i,j} v_j}{\left\| Av_i - \sum_{j=1}^{i} h_{i,j} v_j \right\|_2} \tag{8}$$

With an initial guess x_0, the residual $r_0 = b - Ax_0$ is determined and followed by $v_1 = r_0/\|r_0\|_2$ for the recursion process. The triangular upper Hessenberg matrix, $\overline{H}_m \in \Re^{(m+1)\times m}$, is defined to be H_m plus an extra row $\left(0,\cdots,0,h_{m+1,m}\right)$.

The entries of $\overline{H}_m$, i.e. $h_{i,j}$, used in (8) are given as

$$h_{i,j} = \left(Av_j, v_i\right), 1 \le i \le m+1, 1 \le j \le m \tag{9}$$

If the denominator of (8) becomes zero then a breakdown in GMRES occurs. The vector y_m is calculated by minimising the function $J(y) = \left\|\beta e_1 - \overline{H}_m y\right\|_2$ (see [5]), where $\beta = \|r_0\|_2$ and e_j denotes column j of the identity matrix. The minimiser y_m is inexpensive to compute since it requires the solution of a $(m+1)\times m$ least-square problem given below, where m is typically small [6].

$$\overline{H}_m^T \overline{H}_m y_m = H_m^T e_1 \tag{10}$$

The GMRES method may not converge after m iterations; in this case, a restarted version in which set $x_0 = x_m$ is used with similar sequences described above. A variety of the GMRES formulations used depends on special applications or requirements. The preconditioned GMRES method, from the TEMPLATES package [7], has been used in this study. This algorithm for GMRES with Left Preconditioning is presented here [1].

Algorithm 1. GMRES with left preconditioning

1. Compute $r_0 = M^{-1}(b - Ax_0)$, $\beta = \|r_0\|_2$ and $v_1 = r_0/\beta$
2. For $j = 1, \ldots, m$ Do:
3.   Compute $\omega := M^{-1}Av_j$
4.   For $i = 1, \ldots, j$ Do:
5. $h_{i,j} := (\omega, v_i)$
6. $\omega := \omega - h_{i,j}v_i$
7.   EndDo
8.   Compute $h_{i,j} = \|\omega\|_2$ and $v_{j+1} = \omega/h_{i,j}$
9. EndDo
10. Define $V_m := [v_1, \ldots, v_m]$, $\bar{H}_m = \{h_{i,j}\}$ $1 \le i \le j+1;\ 1 \le j \le m$
11. Compute $y_m = \text{argmin}_y\|\beta e_1 - \bar{H}_m y\|_2$, and $x_m = x_0 + V_m y_m$
12. If convergence satisfied $Stop$, else set $x_0 := x_m$ and $GoTo$ 1

The symbol ":=", used above, means that the right side phrase is substituted into the left parameter.

4. QMR ITERATIVE METHOD

The Quasi-Minimal Residual (QMR) approximates the solution $x = x_0 + V_m y_m$ from the m-th Krylov subspace K_m, where y_m minimises the function $J(y) = \left\|\beta e_1 - \overline{T}y\right\|_2$ just as in GMRES, except that the Arnoldi orthogonal process is replaced by the Lanczos bi-orthogonal process. Additionally, QMR uses look-ahead techniques to avoid breakdowns in the underlying Lanczos process. The Lanczos matrix, T_m, is obtained by replacing the coefficients $\alpha_j \equiv h_{j,j}$ and $\beta_j \equiv h_{j-1,j}$ inside the Hessenberg matrix H_m. The result of the Lanczos algorithm is a relation of the form

$$AV_m = V_{m+1}\overline{T}_m \tag{11}$$

in which $\overline{T}_m$ is the $(m+1)\times m$ tridiagonal matrix constructed from T_m, with a supplementary $(m+1)$ row as $\delta_{m+1}e_m^T$. A full algorithm of QMR with a preconditioner and without a look-ahead is given in TEMPLATE [7]. For an advanced version of QMR with look-ahead implementation, the interested reader is referred to QMRPACK [8].

5. PRECONDITIONING TECHNIQUES

Considering the Krylov subspace, if a good approximate solution cannot be computed easily from this space then a modified problem may be pursued using a preconditioner M, so that the modified system can be solved more effectively. In this case, the system $Ax = b$ is changed to

$$M^{-1}Ax = M^{-1}b \tag{12}$$

where the approximate solutions $x_1, x_2, \cdots, x_k$ will satisfy

$$x_k \in \text{Span}\left\{ \begin{array}{l} M^{-1}b, \left(M^{-1}A\right)M^{-1}b, \cdots, \\ \left(M^{-1}A\right)^{k-1} M^{-1}b \end{array} \right\} \tag{13}$$

at each step of a preconditioned algorithm. It is necessary to compute the product of M^{-1} with a vector, or equivalently, to solve a linear system with the coefficient matrix *M.* Thus, *M* should be chosen so that the resulted linear system can be solved faster than the original one.

Elman and Silvester [9] have shown that the discretised incompressible Navier-Stokes equations may lead to the matrix problem of

$$\begin{pmatrix} F & B^t \\ B & -\beta S \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \tag{14}$$

where *u* and *p* are the discrete velocity vector and the pressure, respectively. *F* is a discrete vector embrace convection-diffusion operator and *B* presents the coupling between the discrete velocity *u* and the pressure *p*. F is also a non-symmetric matrix of the form

$$F = \nu A + N \tag{15}$$

where A is the discrete Laplacian for the Galerkin discretisation and ν is the kinematics viscosity. N is a skew-symmetric matrix corresponding to the discrete convection operator.

a) Stokes preconditioner

A symmetrical matrix can be obtained for preconditioning incompressible Navier-Stokes equations by eliminating the convection terms in (14). This is called Stokes preconditioner given by

$$\begin{pmatrix} \nu A & B^t \\ B & -\beta S \end{pmatrix} \tag{16}$$

An analysis for Stokes preconditioner matrix indicates that this preconditioner is independent of the mesh size [9]. However, this feature does not affirm that the asymptotic convergence rate of GMRES will also be independent of the mesh size.

b) Elman-silvester block triangular preconditioner

The Elman-Silvester block triangular preconditioning matrix is given by

$$\begin{pmatrix} F & B^t \\ 0 & -\frac{1}{\nu}Q \end{pmatrix} \tag{17}$$

where $Q = \left(\Delta h\right)_{ij}^2$ is a function of the mesh size operating in clustered regions. The analysis of this preconditioner has also proven that the convergence of this method is independent of the mesh size. Additionally, the inverse of this block triangular matrix can be expressed in factored form given by

$$\begin{pmatrix} F & B^t \\ 0 & -\frac{1}{\nu}Q \end{pmatrix}^{-1} = \begin{pmatrix} F^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & B^t \\ 0 & -I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \nu Q^{-1} \end{pmatrix} \tag{18}$$

The preliminary results indicated that this preconditioner is an effective tool for the Oseen problem [9].

c) ILU type preconditioners

A complete set of ILU(p) preconditioners were adopted from SPARSKIT2 [5]. The modified version of these techniques, i.e. MLIU, can be obtained by reducing the effects of dropping by a diagonal compensation strategy [1].

6. NUMERICAL TEST CASE

A channel flow over a square bump sketched in Fig. 1 was considered in this study. Reynolds numbers up to 100 were investigated. Three sets of Cartesian mesh were investigated as listed in Table 1. The coarse size and the moderate size meshes are shown in Fig. 2.

Table 1. Three sets of finite element meshes

Mesh set number	Number of elements	Number of nodes	Matrix size (n)
1	48	227	520
2	384	1645	3729
3	600	2541	5753

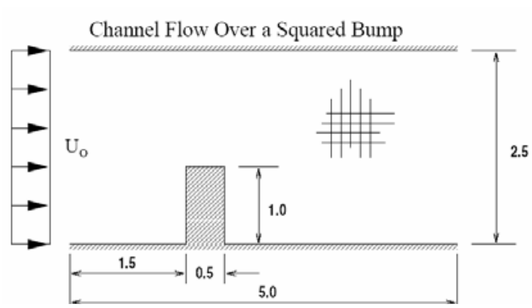

Fig. 1. Channel flow over a square bump

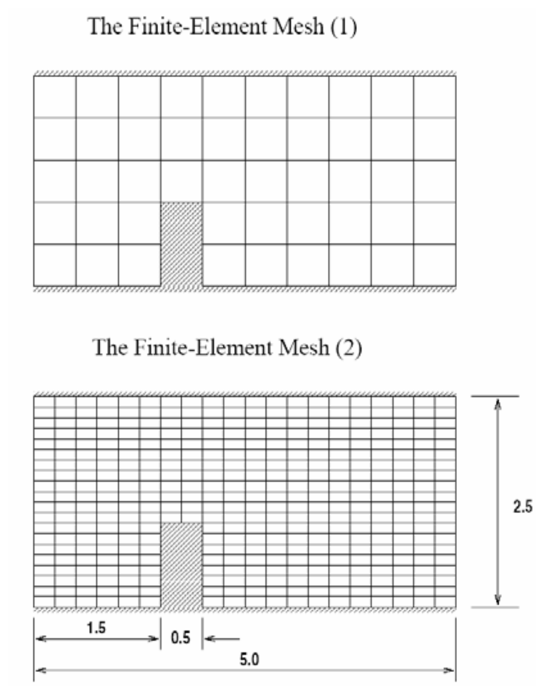

Fig. 2. The coarse size and the medium size finite-element meshes

Each element consists of nine node points, four in the corners, four in the mid-edges, and one in the centre. The degree of freedom of each node point is defined by the number of flow variables related to the particular node point. For instance, each corner point has three degrees of freedom (D.O.F) corresponding to the flow variables *(u, p, v)*, while the rest of the node points have two D.O.F *(u, v)*. Then, one expects 22 unknowns for each element from which 4 unknowns correspond to the pressure (see Fig. 3).

The initial investigations were performed by extracting the full assembled matrix from the FENST-2D code [10], which is a 2D Finite Element code for solving incompressible Navier-Stokes equations for

Sediment Transport and Turbulent flows. For the sediment transport modelling, not a multi-phase, but a continuous phase (or mixture) approach is employed. This implies that the hydrodynamics is solved for the mixture of water and suspended sediment.

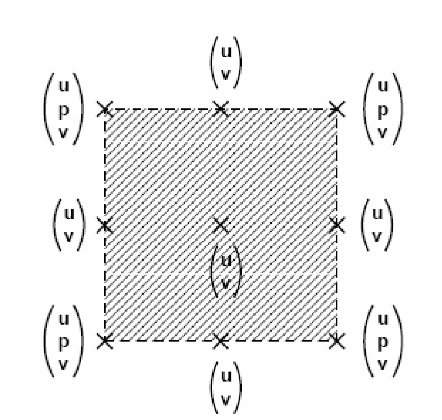

Fig. 3. Element assembly of unknowns

FENST solves a set of equations consisting of the continuity equation for an incompressible fluid and a generalized Navier-Stokes equation which deals with a variable viscosity in the case of turbulent or non-Newtonian laminar flows. The Boussinesq approximation is not employed to account for momentum transfer due to density changes.

FENST uses the two-equation k-epsilon model which solves the conservation of turbulent kinetic energy k and the conservation of energy dissipation rate, plus a module for free surface movement.

Advection-dominated problems may show spurious oscillations as soon as the Peclet number (the element Reynolds number) exceeds the value 1. Therefore, standard and modified upwind stabilization techniques have been implemented, i.e. the standard Petrov-Galerkin method [11], referred to as U_1, and various versions of the Streamline-Upwind/PG method [12], referred to as U_2 and U_3 in this text. The case without upwinding is noted by U_0. Since the kinematics condition for the free surface is a pure advection equation, upwinding is necessary for this equation under all circumstances.

The high degree of the non-linearity of the k-epsilon equations makes it difficult to obtain monotonous convergence of these equations. Stabilization is obtained using two methods: by adding self-eliminating artificial diffusion (SEAD) and by applying a pseudo-time stepping procedure [13]. The equations are solved implicitly and fully coupled. A first-order implicit finite difference discretisation in time is implemented. Further details about the finite-element code and the upwinding methods can be found in [10, 14].

7. RESULTS AND DISCUSSIONS

Non-symmetrical and indefinite matrices were obtained at the fifth iterations from the finite-element code for the test problem in Fig. 1. The medium-sized mesh and a Reynolds number of 100 was used for all comparisons in Figs. 4 to 7. The resulted system eigen-spectrum is shown in Fig. 4 for U_0 and U_1, and in Fig. 5 for U_2 and U_3, respectively. The resulted eigen-spectrum structures of the preconditioned matrices are also shown in Figs. 6 and 7.

In Figs. 6a and 6b, the eigen-spectrum structure of two incomplete factorisation preconditioning matrices, i.e. $M_1^{-1}A$ and $M_2^{-1}A$, are shown for ILU(0) and MILU, respectively. The large negative eigenvalues represent the poor quality of these preconditioners which suggest considering alternative preconditioning techniques. Similarly, the eigen-spectrum structure of Stokes and Elman-Silvester preconditioning matrices are illustrated in Figs. 7a and 7b, respectively. From the eigenspectrum of the resulted matrix, i.e. $M_2^{-1}A,$ it is observed that the Elman-Silvester preconditioning matrix has fewer outer layers with greater concentration near unity, and therefore it is expected to provide a better convergence.

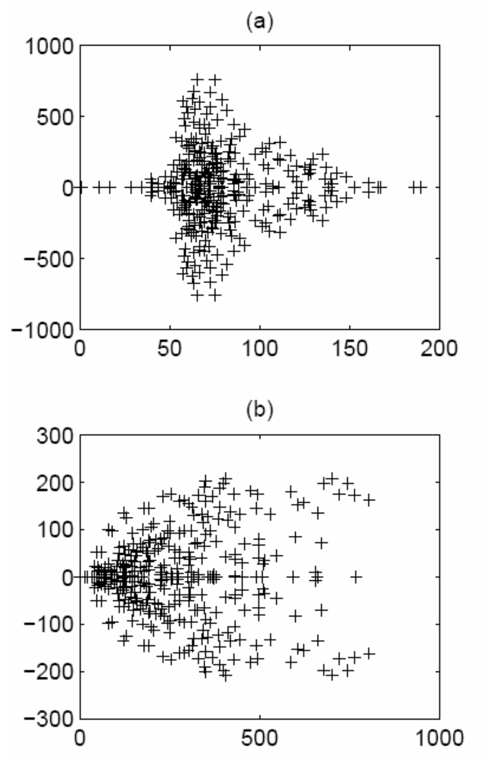

Fig. 4. System eigen-spectrum using the upwind methods: (a) U_0, and (b) U_1

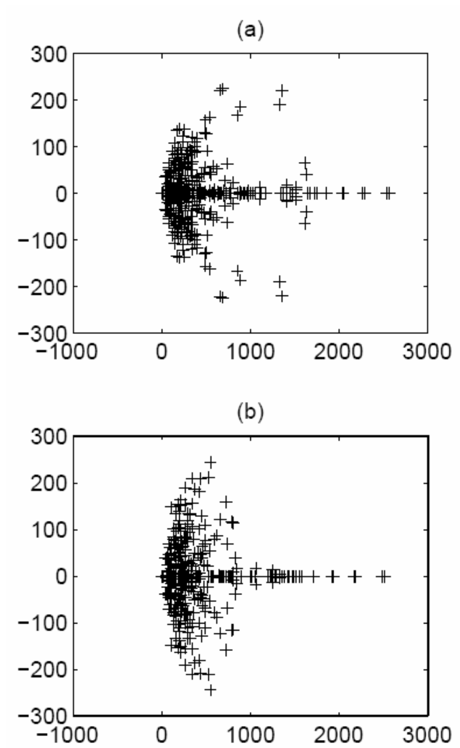

Fig. 5. System eigen-spectrum using the upwind methods: (a) U_2, and (b) U_3

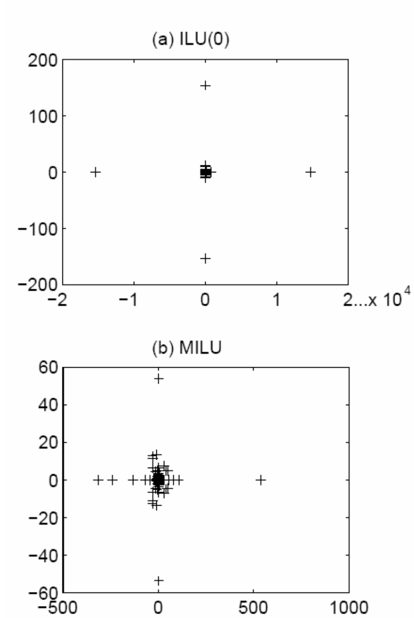

Fig. 6. System eigen-spectrum using preconditioning matrices:
(a) ILU(0), and (b) MILU

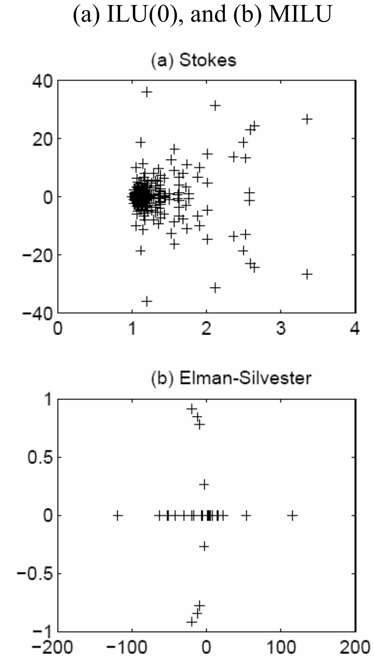

Fig. 7. System eigen-spectrum using preconditioning methods: (a) Stokes,
and (b) Elman-Silvester

The GMRES and QMR iterative solvers without preconditioning were examined for solving the test case using mesh (1). The QMR without preconditioning and also with ILU(0) and MILU preconditioning was diverged. However, GMRES(400), which uses a restart after 400 iterations, with ILU(0), MILU, and without preconditioning was converged to a residual of 10^{-6}, as shown in Fig. 8. In general, from Fig. 8 it is observed that GMRES is sensitive to the choice of upwinding and performed better when using ILU(0) in conjunction with the upwinding method U1. The results obtained using GMRES and QMR, together with Stokes and Elman-Silvester preconditioning techniques, are shown in Fig. 9. Rapid convergence is achieved for both solvers when the Elman-Silvester preconditioning technique was employed. Effects of using different mesh sizes are shown in Fig. 10, which indicates dependency of the convergence rates on mesh sizes. However, these convergence rates were independent from the choice of Reynolds numbers (within the ranges studied), which is of great importance when dealing with turbulent high Reynolds number flows (see Fig. 11).

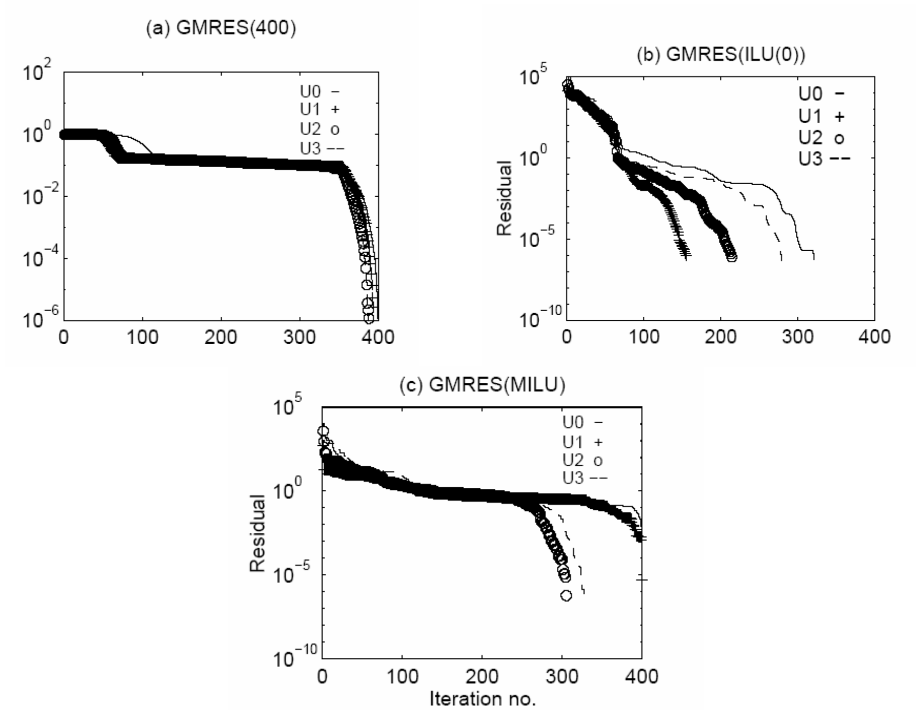

Fig. 8. Convergence rates of (a) GMRES(400) without preconditioner, (b) GMRES with ILU(0), and (c) GMRES with MILU

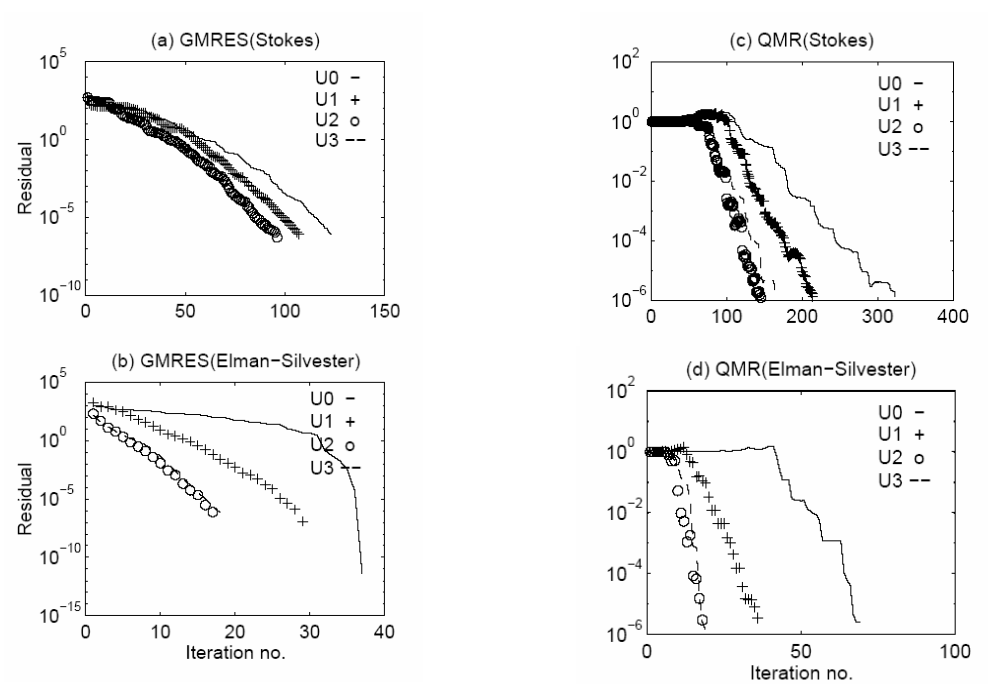

Fig. 9. Convergence rates of GMRES (a, and b) and QMR (c, and d) solvers with Stokes, and Elman-Silvester preconditioning techniques

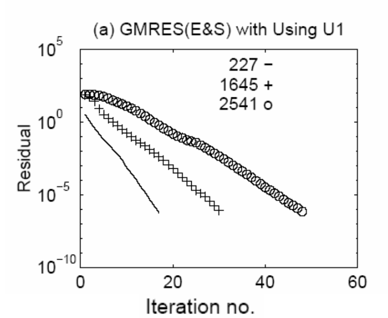

Fig. 10. Mesh size dependency for the convergence rates of GMRES with Elman-Silvester preconditioning technique

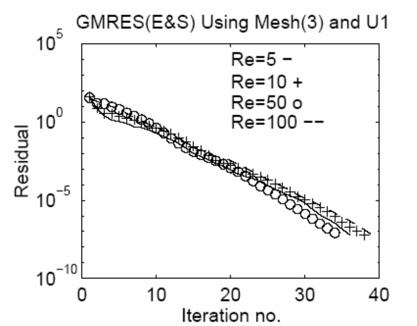

Fig. 11. Reynolds number dependency for the convergence rates of GMRES with Elman-Silvester preconditioning technique

8. CONCLUSION

The linear system of equations derived from the finite-element discretisation of the incompressible Navier-Stokes equations for hydrodynamics problems is usually a large sparse system. It is argued that the memory and the computational requirements for solving two- or three- dimensional PDE's, involving many degrees of freedom per point, may seriously challenge the most efficient direct solvers (such as the Frontal approach) available today.

In this work, two Krylov subspace iterative methods, GMRES and QMR, were studied in conjunction with several preconditioning techniques including the incomplete factorisation methods such as ILU(0) and MILU, the Stokes preconditioner, and the Elman-Silvester block triangular preconditioner. For the test cases studied, it is observed that the GMRES solver with the Elman-Silvester preconditioner provides faster convergence than the other methods. Although the solver is sensitive to the size of the meshes and the choice of upwinding, it is interesting to point out that the GMRES solver with the Elman-Silvester preconditioner produced the same convergence rate for a range of Reynolds numbers considered. This is particularly significant when dealing with high Reynolds number turbulent flows, which cover a wide range of practical problems.

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