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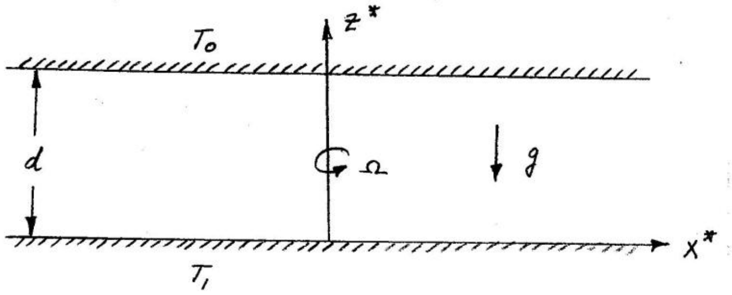
Problem Set 5

Problem 1 (encompasses Drazin & Reid 2.3 and 2.10)

Consider thermal convection of a Boussinesq fluid uniformly rotating at angular velocity $\Omega\mathbf{k}$ in the physical situation sketched below, where asterisk variables are dimensional.



Part A

For the same scalings of time, space, velocity, pressure and temperature used in class, obtain the appropriate set of nondimensional equations and boundary conditions that involve the following three parameters

$$Ra = \frac{g\alpha\Delta T d^3}{\kappa\nu} \qquad Pr = \frac{\nu}{\kappa} \qquad Ta = \frac{4\Omega^2 d^4}{\nu^2}.$$

Derive the equations describing the evolution of infintesimal disturbances. Again take the curl of the momentum equation twice to show that the governing equations are

$$\frac{\partial\theta'}{\partial t} = \nabla^2\theta' + w' \tag{1}$$

$$\frac{\partial\omega_3'}{\partial t} - Pr\sqrt{Ta}\frac{\partial w'}{\partial z} = Pr\nabla^2\omega_3' \tag{2}$$

$$\frac{\partial}{\partial t}\nabla^2 w' + Pr\sqrt{Ta}\frac{\partial\omega_3'}{\partial z} = RaPr\nabla_H^2\theta' + Pr\nabla^4 w' \tag{3}$$

where ω_3' is the vertical component of vorticity. Now use normal modes

$$\begin{pmatrix} w' \\ \theta' \\ \omega_3' \end{pmatrix} = \begin{pmatrix} W(z) \\ \theta(z) \\ \zeta(z) \end{pmatrix} f(x,y)e^{st}$$

to obtain separable equations governing the three vertical eigenfunctions. Finally, eliminate $\theta(z)$ and $\zeta(z)$ to obtain a single equation for $W(z)$. (Whenever possible adopt results already derived in class.)

Part B

To make computations easy, we now assume only stress-free horizontal surfaces. Derive two equations governing $\theta(z)$ and $W(z)$ and, following the procedure in class for finding positive definite integrals, obtain the general result

$$K_1+K_2(\sigma+i\omega)+K_3(\sigma^2-\omega^2+2i\sigma\omega) = Ra[K_4+K_5(\sigma+i\omega)+K_6(\sigma-i\omega)+K_7(\sigma^2+\omega^2)] \quad (4)$$

where the K_n are positive definite integrals. Deduce from the imaginary part that one possible solution is $\omega = 0$ corresponding to exchange of stability. For this case analyze the roots of σ from the real part to show that for exchange of stability one must have $Ra > 0$ for sustained convection of infintesimal disturbances.

Part C

Again considering both surfaces to be stress-free, show that the solution $\sin(n\pi z)$ satisfies the boundary-value problem for $W(z)$ and *all* its boundary conditons — you will have to find two more boundary conditions on $W(z)$ since it is now an eighth order equation. By analyzing the eigenvalue relation $Ra = Ra(s; Ta, Pr)$ derived for this case, show that there will be modes for which exchange of stability is not valid (*i.e.*, $\omega \neq 0$ at $\sigma = 0$) when

$$Ta > \frac{(1+Pr)\pi^4}{1-Pr}.$$

Note: I think the result given in D & R 2.10 in both the old and new edition is incorrect. The denominator for Ta should be $(1-Pr)(j\pi)^2$ and not $(1-Pr)(j\pi)^3$.