JAST, Vol. 3, No. 3, pp 1–8
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JAST
Journal of Aerospace Science and Technology

Normal Form Solution of Reduced Order Oscillating Systems

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This paper describes a preliminary investigation into the use of normal form theory for modelling large non-linear dynamic systems. Limit cycle oscillations are determined for simple two-degree-of-freedom double pendulum systems. Such a system is reduced into its centre manifold before computation of normal forms, which are obtained using a period averaging method applicable to non-autonomous systems and more advantageous than the classical methods. A good agreement was observed between the predicted results from the normal form theory and the numerical simulations of the original system.

NOMENCLATURE

A	Jacobian matrix
$a_i, i = 1, ..., 4$	constant parameters, see Eq. 17
$a_{ij}, i, j = 0, ..., 3$	constants, see Eq. 30
B	Canonical matrix
$b_{ij}, i, j = 0, ..., 3$	constants, see Eq. 30
D	dissipation function
D_u	partial derivative with respect to u
D_v	partial derivative with respect to v
$D_{ij}, i, j = 1, 2$	elements of the Jacobian matrix, see Eq. 32
d	damping coefficient, see Eq. 14
$f(x, \epsilon)$	n-vector nonlinear function
$f_k^0(\zeta)$	kth order normal forms
$f_i, i = 1, ..., 4$	non-dimensional quantities, see Eq. 14
G_u, G_v	third order functions
$g_k(\zeta, t)$	kth order periodic transformation function
$g(z)$	nonlinear function of the state variable z
$g(x)$	nonlinear function of the transformed variable x
$h_k(\zeta, t)$	kth order geometrical transformation function
$h'_k(\zeta, k)$	derivative with respect to ζ
$h_i, i = 1, 2$	system coordinates, see Eq. 30
k_1, K_2, k_3	system stiffness coefficients for linear springs
J	Jordan canonical matrix
J_1	matrix corresponding to critical mode of the system
J_2	matrix corresponding to remainder
l	length of the weightless links, see Eqs. 10 and 11
m	mass, see Eqs. 10 and 14
P	follower force
$P(\Lambda)$	characteristic polynomial
Q_i	generalized force in the i-direction
q_i	generalized coordinates in the i-direction
T	kinetic energy
u	critical mode coordinates
v	coordinates corresponding to remainder
V	potential energy
x	state vector
$\dot{x}$	time derivative of x
$x_i, i = 1, ..., 4$	components of the vector x
y	vector with n components
$\dot{y}$	time derivative of y
$z_i, i = 1, ..., 4$	state variables
$\dot{z}$	time derivative of z

Greeks

ϵ	small perturbation parameter

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η	non-dimensional system indicator, see Eq. 7
λ	characteristic variable
$\lambda_i, i = 1, ..., 4$	eigenvalues
μ	transformed characteristic variable
μ_c	critical value of the characteristic variable
θ_1, θ_2	system configuration coordinates
ζ	n-component vector
$\dot{\zeta}$	time derivative of ζ

INTRODUCTION

Vibration behaviour such as Limit Cycle Oscillations (LCO) can only occur in non-linear systems [1,2]. Consequently, it is not possible to predict LCO using a purely linear analysis. Moreover, linear analysis is becoming less feasible. LCO has become an important research topic over the last few years although such problems have been encountered long ago.

A periodic solution of a dynamical system is called a limit cycle if there are no other periodic solutions sufficiently close to it. In other words, a limit cycle is an isolated periodic solution and corresponds to an isolated closed orbit in the state space [3]. Every trajectory initiated near a stable limit cycle approaches it as $t \rightarrow \infty$.

Prediction of limit cycle oscillations (LCO) has been carried out for a range of simple non-linear dynamical systems [4-17] using normal form theory (NFT). Only recently, has computation of normal forms for general M-DOF systems using multiple time scales been reported [18, 19]. The NFT is used to simplify analytical expressions for non-linear systems [5-6]. In this method, a non-linear co-ordinate transformation is employed to obtain a simple analytical expression for the transformed equations such that qualitative behaviour of the system is evaluated without solving the system of equations. The classical approach of Poincare [20] and Brikhoff [21] suffers from evaluating large matrices to obtain normal forms. Liu [22] and Grzedzinski [23] have applied center manifold theory to reduce the number of differential equations before computing normal forms. Zhang [9] has calculated normal forms through a period averaging method that can be used for solving the governing equations of non-autonomous systems.

In this paper, the method developed by Zhang [13] was adopted for solving LCOs for a two-degrees-of-freedom double pendulum system. A non-linear system is reduced into its critical modes (or center manifold) which may correspond to one or two single DOF systems. The reduced system exhibits exactly the same behaviour as the full system at the corresponding modes for which higher order normal forms can be obtained with less computational effort. The methodology described involves transformation of system equations into modal canonical forms, reduction of these equations into normal forms, and then prediction of instability behaviour, here LCO. Predictions are verified through comparisons with numerical simulations.

PERIOD AVERAGING METHOD

It is computationally exhaustive to find coefficients of normal forms using the matrix approach [5, 6]. An alternative faster approach is the averaging method [7-8] which is equivalent to the NFT method. Thus, the problem of calculating higher order coefficients of normal forms is equivalent to the problem of calculating higher order averaging equations. Since the averaging method is applied to non-autonomous systems, a coordinate transformation is adopted such that an autonomous system is obtained through a time integrating procedure.

In this approach, we have the following non-linear ordinary differential equation, where is a perturbation parameter,

$$\dot{x} = Jx + \epsilon f\left(x, \epsilon\right), \qquad x \in \Omega \in R^n \tag{1}$$

is transformed to a time dependent ordinary differential equation in y,

$$\dot{y} = \epsilon e^{-t\,J} f\left(e^{t\,J} y, \epsilon\right) = \epsilon g\left(y, t, \epsilon\right) \tag{2}$$

using the transformation function

$$x = e^{t\,J} y \text{ and } \dot{x} = J e^{t\,J} y + e^{t\,J} \dot{y}, \tag{3}$$

where $0 < |x| << 1$, $f \in C^{r+1}$ and $f\left(0, \epsilon\right)$ =0. Here, J is the Jordan canonical matrix, Ω is a domain which contains the origin and is invariant under Γ, $\Gamma x \in \Omega$ if $x \in \Omega$. Note that equation (2) explicitly depends on time while the original equation (1) does not.

The period averaged normal form of equation (2) can be constructed using the following change of variable:

$$y = \zeta + \sum_{l=1}^{m} \epsilon^l h_l\left(\zeta, t\right), \tag{4}$$

which transforms equation (2) to a normal form up to the order m as follows:

$$\dot{\zeta} = \sum_{k=1}^{m} \epsilon^k f_k^0\left(\zeta\right) + O\left(\epsilon^{m+1}\right), \tag{5}$$

where the geometrical transformations $h_k\left(\zeta, t\right)$ in equation (4) are given by

$$h_k\left(\zeta, t\right) = \frac{1}{T}\int_0^T \tau\left[g_k\left(\zeta, \tau + t\right) - f_k^0\right]\, d\tau. \tag{6}$$

The normal forms $f_k^0(\zeta)$ are given by

$$f_k^0(\zeta) = \frac{1}{T}\int_0^T g_k(\zeta,\tau)\,d\tau, \tag{7}$$

and the functions $g_k(\zeta,t)$ is determined by

$$g_k(\zeta,t) = \frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial\epsilon^{k-1}} g\left(\left(\zeta + \sum_{l=1}^{k-1}\epsilon^l h_l(\zeta,t)\right), t, \epsilon\right)_{\epsilon=0} - \sum_{l=1}^{k-1} h'_{k-1}(\zeta,t)\, f_l^0(\zeta), \tag{8}$$

where a prime denotes differentiation with respect to ζ. More details on deriving the above relationships can be found in [28].

DOUBLE PENDULUM SYSTEM

Double pendulum system shown in Figure 1 consists of two rigid weightless links of equal length l, which carry two concentrated masses 2m and m, respectively.

A follower force P is applied to this system. Equations of motion can be obtained for this system using the Lagrange's equation [24-25]:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \left(\frac{\partial D}{\partial \dot{q}_i}\right) - \left(\frac{\partial T}{\partial q_i}\right) + \left(\frac{\partial V}{\partial q_i}\right) = Q_i, \quad i = 1,2. \tag{9}$$

where T is the kinetic energy; D is the dissipation function; V is the potential function; and $Q_i = \partial(\delta W)/\partial(\delta q_i)$ is the ith generalized force with δW being the virtual work done by the force Q_i. The virtual displacement at the exertion point of Q_i is δq_i.

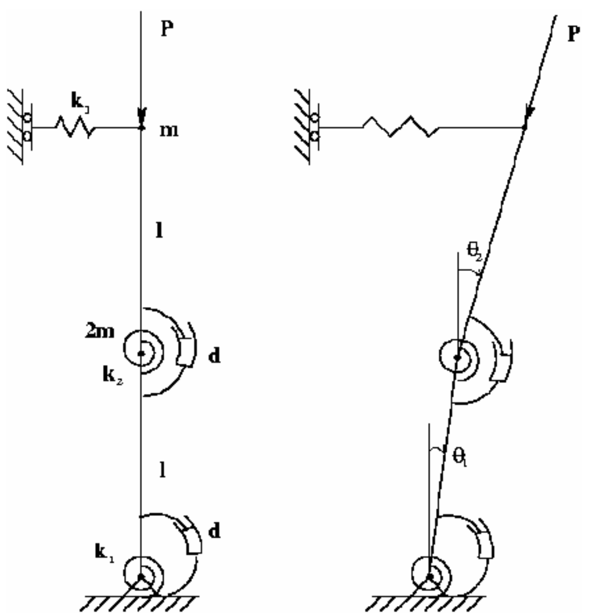

Figure 1. A sketch of a double pendulum system with follower force.

Considering the above system, the kinetic energy T becomes [26],

$$T = \frac{ml^2}{2\Omega^2}\left[3\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right], \tag{10}$$

where θ_1 and θ_2 are generalized coordinates that define the configuration of the system. The potential energy associated with three linear springs k_1, k_2 and k_3 is given by [26]:

$$V = \frac{1}{2}\left[\left(k_1+k_2+k_3l^2\right)\theta_1^2 + 2\left(k_3l^2-k_2\right)\theta_1\theta_2 + \left(k_2+k_3l^2\right)\theta_2^2\right] - \frac{1}{6}k_3l^2(\theta_1+\theta_2)\left(\theta_1^3+\theta_2^3\right). \tag{11}$$

Lagrange's equations introduced in equation (9) lead to a set of first order differential equations as follows [26],

$$\begin{aligned}
\frac{dz_1}{dt'} &= z_2,\\
\frac{dz_2}{dt'} &= -\frac{1}{2}(f_1+2f_2-\eta)\,z_1 - \frac{3}{2}f_4z_2\\
&\quad + \frac{1}{2}(2f_2-\eta)\,z_3 + f_4z_4\\
&\quad + \frac{1}{12}(3f_1+9f_2+2f_3-4\eta)\,z_1^3\\
&\quad - \frac{1}{4}(2f_1+9f_2-2f_3-4\eta)\,z_1^2z_3\\
&\quad + \frac{1}{4}(f_1+9f_2-4\eta)\,z_1z_3^2\\
&\quad - \frac{1}{6}(3f_2+3f_3-2\eta)\,z_3^3\\
&\quad + \frac{1}{4}f_3(3z_2-2z_4)(z_1-z_3)^2\\
&\quad - z_2z_4(z_1-z_3),
\end{aligned} \tag{12a}$$

$$\begin{aligned}
\frac{dz_3}{dt'} &= z_4,\\
\frac{dz_4}{dt'} &= +\frac{1}{2}(f_1+4f_2-2f_3-\eta)\,z_1 + \frac{5}{2}f_4z_2\\
&\quad - \frac{1}{2}(4f_2+2f_3-\eta)\,z_3 - 2f_4z_4\\
&\quad - \frac{1}{12}(6f_1+15f_2-2f_3-7\eta)\,z_1^3\\
&\quad + \frac{1}{4}(4f_1+15f_2-f_3-7\eta)\,z_1^2z_3\\
&\quad - \frac{1}{4}(2f_1+15f_2-2f_3-7\eta)\,z_1z_3^2\\
&\quad + \frac{1}{12}(15f_2+14f_3-7\eta)\,z_3^3\\
&\quad - \frac{1}{4}f_4(5z_2-4z_4)(z_1-z_3)^2,
\end{aligned} \tag{12b}$$

where the state variables z_i are defined as:

$$z_1 = \theta_1, \quad z_2 = \dot{\theta}_1, \quad z_3 = \theta_2, \quad z_4 = \dot{\theta}_2, \tag{13}$$

and the non-dimensional quantities f_i and η are given by:

$$f_1 = \frac{k_1\Omega^2}{ml^2}, \quad f_2 = \frac{k_2\Omega^2}{ml^2}, \quad f_3 = \frac{k_3\Omega^2}{m},$$
$$f_4 = \frac{d\Omega^2}{ml^2}, \quad \eta = \frac{P\Omega^2}{ml}. \tag{14}$$

Here f_i (i=1, 2, 3, and 4) are introduced according to physical constraints and η is a system indicator parameter.

The system of equation (12) can be rewritten as:

$$\dot{z} = Az + g(z), \tag{15}$$

where Az is the linear part and g(z) is the non-linear part. The Jacobian matrix A is evaluated at z=0 as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{2}(f_1 + 2f_2 - \eta) & -\frac{3}{2}f_4 & \frac{1}{2}(2f_2 - \eta) & f_4 \\ 0 & 0 & 0 & 1 \\ * & \frac{5}{2}f_4 & ** & -2f_4 \end{bmatrix}$$

$$* = \frac{1}{2}(f_1 + 4f_2 - 2f_3 - \eta)$$
$$** = -\frac{1}{2}(4f_2 + 2f_3 - \eta) \tag{16}$$

from which one may obtain the characteristic polynomial:

$$P(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4, \tag{17}$$

with

$$a_1 = \frac{7}{2}f_4,$$
$$a_2 = \frac{1}{2}\left(f_4^2 + f_1 + 6f_2 + 2f_3 - 2\eta\right),$$
$$a_3 = \frac{1}{2}(f_1 + f_2 + 5f_3)f_4,$$
$$a_4 = \frac{1}{2}f_1(f_2 + f_3) + f_3(2f_2 - \eta). \tag{18}$$

It can be shown that at the critical point defined by:

$$f_1 = \frac{1}{2}, \quad f_2 = \frac{5}{8}, \quad f_3 = \frac{1}{8}, \quad f_4 = \frac{1}{2}, \quad \eta_c = \frac{3}{2}. \tag{19}$$

the polynomial $P(\lambda)$ has a pair of purely imaginary but distinct negative eigenvalues:

$$\lambda_{1,2} = \pm\frac{1}{2}i, \qquad \lambda_3 = -\frac{1}{2}, \qquad \lambda_4 = -\frac{5}{4}. \tag{20}$$

Shifting the parameter as:

$$\mu = \eta - \eta_c = \eta - \frac{3}{2}, \tag{21}$$

and transforming the Jacobian matrix A into its modal canonical form [26-27] using the canonical matrix B, i.e. z=Bx,

$$B = \begin{bmatrix} -\frac{1}{20} & \frac{7}{20} & -2 & -1 \\ -\frac{7}{40} & -\frac{1}{40} & 1 & \frac{5}{4} \\ 0 & \frac{1}{2} & 0 & 1 \\ -\frac{1}{4} & 0 & 0 & -\frac{5}{4} \end{bmatrix} \tag{22}$$

One may transform the system (15) into the following system:

$$\dot{x} = Jx + g(x), \tag{23}$$

where the Jacobian canonical matrix J at the origin x_i=0 and at the critical point $\mu_c = 0$ is given by:

$$J = B^{-1}AB = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{5}{4} \end{bmatrix} \tag{24}$$

The non-linear part g(x) is given in the Appendix.

REDUCTION TO CENTER MANIFOLD

Substituting the matrix J from equation (24) and the set of functions $g_{i,j}(x)$ from appendix into equation (23), and changing the time scale into $t = \frac{1}{2}t'$, one may obtain:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$
$$+ 2 \begin{bmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,20} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,20} \\ g_{3,1} & g_{3,2} & \cdots & g_{3,20} \\ g_{4,1} & g_{4,2} & \cdots & g_{4,20} \end{bmatrix} X, \tag{25}$$

where the vector X is defined as:

$$X = [x_1^3, x_2^3, x_3^3, x_4^3, x_1^2x_2, x_1^2x_3, x_1^2x_4, x_2^2x_1, x_2^2x_3, x_2^2x_4, x_3^2x_1, x_3^2x_2, x_3^2x_4, x_4^2x_1, x_4^2x_2, x_4^2x_3, x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4]^T, \tag{26}$$

(The coefficients of the function g(x) denoted by $g_{i,j}$ are provided in the appendix). The above system can be decomposed into two systems of equations as follows:

$$\dot{u} = J_1u + f_1(u,v) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} u + G_u,$$
$$\dot{v} = J_2v + f_2(u,v) = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{5}{2} \end{bmatrix} v + G_v, \tag{27}$$

where

$$G_u = 2\begin{bmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,20} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,20} \end{bmatrix} X,$$

$$G_v = 2\begin{bmatrix} g_{3,1} & g_{3,2} & \cdots & g_{3,20} \\ g_{4,1} & g_{4,2} & \cdots & g_{4,20} \end{bmatrix} X. \tag{28}$$

where Here, $u = (x_1, x_2)$ and $v = (x_3, x_4)$ are decomposed coordinates and $u = (x_1, x_2)$ is assumed to be the corresponding critical mode. We are seeking for a solution near the origin x=(0,0,0,0). If an approximate function $v = h(u)$ is found near the origin so that the following relationship holds

$$\begin{aligned} \dot{v} = D_u h(u)\dot{u} &= D_u h(u)\left[J_1 u + f_1(u, h(u))\right] \\ &= J_2 h(u) + f_2(u, h(u)), \end{aligned} \tag{29}$$

the first equation in (27) will only be needed for investigating the critical mode. In the above relation, the matrix $D_u h(u)$ is the Jacobian of $h(u)$.

NORMAL FORM COEFFICIENTS

The function $h(u)$ can be approximated by a third order function of the form:

$$\begin{aligned} v = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = h(u) = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \\ = \begin{bmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a_{20} & a_{11} & a_{02} \\ b_{20} & b_{11} & b_{02} \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix} \\ + \begin{bmatrix} a_{30} & a_{21} & a_{12} & a_{03} \\ b_{30} & b_{21} & b_{12} & a_{03} \end{bmatrix} \begin{bmatrix} x_1^3 \\ x_1^2 x_2 \\ x_1 x_2^2 \\ x_2^3 \end{bmatrix} \end{aligned} \tag{30}$$

where a_{ij} and b_{ij} are unknown constants. Moreover, the Jacobian matrix, $D_u h(u)$, can be determined as follows:

$$D_u h(u) = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} \end{bmatrix} \tag{31}$$

where

$$\begin{aligned} D_{11} &= a_{10} + 2a_{20}x_1 + a_{11}x_2 + 3a_{30}x_1^2 + 2a_{21}x_1x_2 + a_{12}x_2^2, \\ D_{12} &= a_{01} + 2a_{02}x_2 + a_{11}x_1 + 3a_{03}x_2^2 + 2a_{12}x_1x_2 + a_{21}x_1^2, \\ D_{21} &= b_{10} + 2b_{20}x_1 + b_{11}x_2 + 3b_{30}x_1^2 + 2b_{21}x_1x_2 + b_{12}x_2^2, \\ D_{12} &= b_{01} + 2b_{02}x_2 + b_{11}x_1 + 3b_{03}x_2^2 + 2b_{12}x_1x_2 + b_{21}x_1^2, \end{aligned} \tag{32}$$

Substituting equations (30)-(32) into (29) results in:

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \left\{ \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} + \begin{bmatrix} G_u(1) \\ G_u(2) \end{bmatrix} \right\} = \begin{bmatrix} -h_1 \\ -\frac{5}{2}h_2 \end{bmatrix} + \begin{bmatrix} G_v(1) \\ G_v(2) \end{bmatrix}, \tag{33}$$

where the functions G_u and G_v are third order polynomials of the state variables. Using (30) and (32) in (33) and ignoring all terms of $O(|x^4|)$, a set of algebraic equations are obtained for the unknown coefficients in (30) by equating the coefficients of similar monomials in the left and right hand sides of (33). This set of algebraic equations was solved using an algebraic processor in Mathematica [28]. Thus, the reduced system (up to third order terms) becomes:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \epsilon \begin{bmatrix} c_1x_1^3 + c_2x_1^2x_2 + c_3x_1x_2^2 + c_4x_2^3 \\ c_1'x_1^3 + c_2'x_1^2x_2 + c_3'x_1x_2^2 + c_4'x_2^3 \end{bmatrix}, \tag{34}$$

where

$$[c_1, c_2, c_3, c_4] = \begin{bmatrix} -\frac{83069}{11136000} & -\frac{88487}{3712000} & \frac{231}{128000} & -\frac{931423}{11136000} \end{bmatrix}$$

and

$$[c_1', c_2', c_3', c_4'] = \begin{bmatrix} -\frac{16811}{5568000} & -\frac{18753}{1856000} & -\frac{111}{64000} & -\frac{6179}{1856000} \end{bmatrix}$$

are constants obtained from the programme in Mathematica.

RESULTS AND DISCUSSIONS

The reduced system (34) is shifted from the origin based on the system parameter μ and is then analysed for predicting limit cycle oscillations. The computed centre manifold equation is given as:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{8}{145}\mu x_1 + \frac{24}{145}\mu x_2 \\ \frac{9}{145}\mu x_1 + \frac{27}{145}\mu x_2 \end{bmatrix} \\ + \epsilon \begin{bmatrix} c_1x_1^3 + c_2x_1^2x_2 + c_3x_1x_2^2 + c_4x_2^3 \\ c_1'x_1^3 + c_2'x_1^2x_2 + c_3'x_1x_2^2 + c_4'x_2^3 \end{bmatrix} \end{aligned} \tag{35}$$

Taking $\mu = 0.1$ and $\epsilon = 1$, and solving down to the third order normal forms, the following normalized system of equations is obtained:

$$\begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0.012069r - 0.00523608r^3 \\ -0.00500178 + 0.0314786r^2 \end{bmatrix} \tag{36}$$

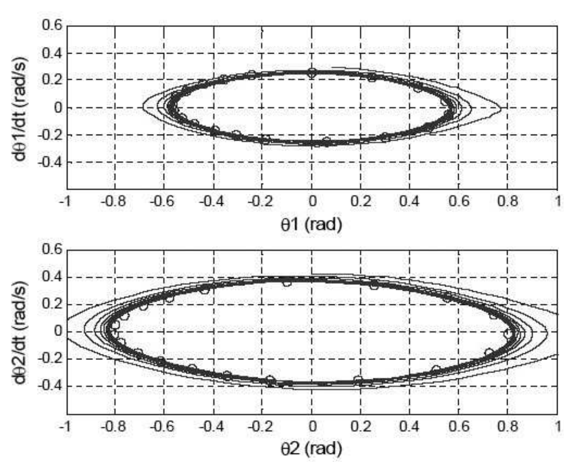

Figure 2. Comparison of analytical NFT (symbols) with numerical Runge-Kutta (solid lines) LCO solutions (IC1).

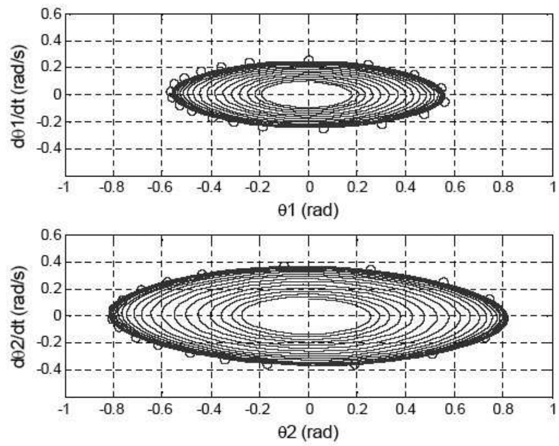

Figure 3. Comparison of analytical NFT (symbols) with numerical Runge-Kutta (solid lines) LCO solutions (IC2).

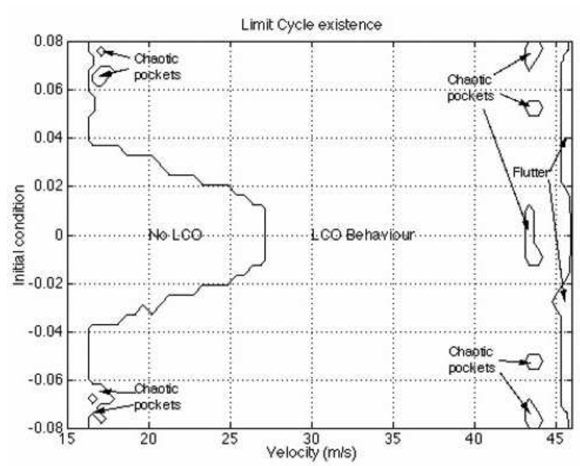

Figure 4. Example of limit cycle oscillation boundaries for a simple non-linear aeroelastic system [16].

The steady state solution of $(r, \theta) = (1.51821, 0.0676t)$ is obtained from (36). Using the steady state solution and reversing the coordinate transformations, the solution of the original system (physical system) for limit cycle oscillations are obtained as plotted in Figures 2 and 3.

The results are compared with the Runge-Kutta numerical solution for two sets of initial conditions. The first initial condition corresponds to a point outside LCO as:

$$\text{IC1:} \ (x_1, x_2, x_3, x_4) = (-1.70, 0, 0, 0) \ \text{or}$$

$$(z_1, z_2, z_3, z_4) = (0.085, 0.2975, 0.0, 0.425)$$

and the second initial condition corresponds to a point inside LCO as:

$$\text{IC2:} \ (x_1, x_2, x_3, x_4) = (-0.5, 0, 0, 0) \ \text{or}$$

$$(z_1, z_2, z_3, z_4) = (0.025, 0.0875, 0.0, 0.125)$$

Numerical results are compared with analytical solutions obtained from normal forms in Figures 2 and 3, for IC1 and IC2 conditions, respectively. For the initial condition outside LCO, the numerical results have converged exactly to the same values obtained by theory as shown in Figure 2. However, there is a mismatch between numerical simulations and analytical results as is seen in Figure 3. It may be argued that numerical simulations corresponding to IC2 require more iterations to assure convergency. Furthermore, normal forms have been obtained up to the third order approximations. In some cases, numerical simulations may diverge due to the irregular behaviour of the dynamical system (see Figure 4). An example of limit cycle oscillation boundaries for a simple non-linear aeroelastic system in Figure 4 shows the initial condition dependency of such systems [16]. This may suggest choosing IC1 as a type of initial condition for faster numerical convergence.

CONCLUSION

In this study, we have carried out a limit cycle analysis of a two-degrees-of-freedom nonlinear double pendulum system. The order of the system is reduced by a center manifold approach corresponding to the critical mode of the system. Normal forms were successfully obtained by a period averaging method for the reduced system. Normal forms and limit cycle oscillations were obtained for the reduced system using a symbolic programming code in Mathematica. The numerical simulations for the full system using the Runge-Kutta method were compared with LCO solutions obtained from the analytical approach. The analytical normal form estimations are in good agreement with the numerical results. We have not encountered notable difficulties while using Mathematica; however, longer run-time and larger machine memory are required for higher order dynamic systems. Further research is needed for developing a general reduction code, preferably in non symbolic operating environment, for dealing with higher-order dynamic systems.

ACKNOWLEDGMENT

The author wishes to acknowledge kind assistance of Dr M. Hemami on proof reading of this manuscript from the Department of Mechanical Engineering at Isfahan University of Technology.

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APPENDIX 1.

$$
\begin{aligned}
g_1 &= [g_{1,1} \quad g_{1,2} \quad \dots \quad g_{1,20}]\, X \\
&= -\frac{83069}{11136000}x_1^3 - \frac{931423}{11136000}x_2^3 + \frac{631}{174}x_3^3 + \frac{1318}{87}x_4^3 \\
&\quad - \frac{88487}{3712000}x_1^2x_2 - \frac{23169}{92800}x_1^2x_3 - \frac{102613}{3712000}x_1^2x_4 \\
&\quad + \frac{231}{128000}x_1x_2^2 + \frac{17799}{92800}x_2^2x_3 - \frac{125837}{3712000}x_2^2x_4 \\
&\quad + \frac{4731}{2320}x_1x_3^2 - \frac{1087}{2320}x_2x_3^2 + \frac{4587}{232}x_3^2x_4 + \frac{1185}{928}x_1x_4^2 \\
&\quad + \frac{305}{928}x_2x_4^2 + \frac{3725}{116}x_3x_4^2 + \frac{3213}{46400}x_1x_2x_3 \\
&\quad + \frac{3041}{185600}x_1x_2x_4 + \frac{14987}{46400}x_1x_3x_4 + \frac{3841}{4640}x_2x_3x_4,
\end{aligned}
$$

$$
\begin{aligned}
g_2 &= [g_{2,1} \quad g_{2,2} \quad \dots \quad g_{2,20}]\, X \\
&= -\frac{16811}{5568000}x_1^3 - \frac{6179}{1856000}x_2^3 - \frac{71}{87}x_3^3 - \frac{221}{87}x_4^3 \\
&- \frac{18753}{1856000}x_1^2x_2 - \frac{4831}{46400}x_1^2x_3 - \frac{22077}{1856000}x_1^2x_4 \\
&- \frac{111}{64000}x_1x_2^2 - \frac{999}{46400}x_2^2x_3 - \frac{13173}{185600}x_2^2x_4 \\
&+ \frac{749}{1160}x_1x_3^2 - \frac{1273}{1160}x_2x_3^2 - \frac{637}{116}x_3^2x_4 + \frac{5}{464}x_1x_4^2 \\
&- \frac{635}{464}x_2x_4^2 - \frac{415}{58}x_3x_4^2 - \frac{413}{23200}x_1x_2x_3 \\
&- \frac{5911}{92800}x_1x_2x_4 + \frac{1443}{2320}x_1x_3x_4 - \frac{5751}{2320}x_2x_3x_4,
\end{aligned}
$$

$$
\begin{aligned}
g_3 &= [g_{3,1} \quad g_{3,2} \quad \dots \quad g_{3,20}]\, X \\
&= -\frac{1123}{1024000}x_1^3 + \frac{2077}{3072000}x_2^3 - \frac{7}{16}x_3^3 - \frac{35}{24}x_4^3 \\
&- \frac{3787}{1024000}x_1^2x_2 - \frac{973}{25600}x_1^2x_3 - \frac{4469}{102400}x_1^2x_4 \\
&- \frac{801}{1024000}x_1x_2^2 - \frac{357}{25600}x_2^2x_3 - \frac{2221}{102400}x_2^2x_4 \\
&+ \frac{143}{640}x_1x_3^2 - \frac{291}{640}x_2x_3^2 - \frac{181}{64}x_3^2x_4 - \frac{7}{256}x_1x_4^2 \\
&- \frac{151}{256}x_2x_4^2 - \frac{123}{32}x_3x_4^2 - \frac{119}{12800}x_1x_2x_3 \\
&- \frac{1407}{51200}x_1x_2x_4 + \frac{47}{256}x_1x_3x_4 - \frac{275}{256}x_2x_3x_4,
\end{aligned}
$$

$$
\begin{aligned}
g_4 &= [g_{4,1} \quad g_{4,2} \quad \dots \quad g_{4,20}]\, X \\
&= \frac{16811}{11136000}x_1^3 + \frac{6179}{3712000}x_2^3 + \frac{71}{174}x_3^3 + \frac{221}{174}x_4^3 \\
&+ \frac{18753}{3712000}x_1^2x_2 - \frac{4831}{92800}x_1^2x_3 + \frac{22077}{371200}x_1^2x_4 \\
&+ \frac{111}{128000}x_1x_2^2 + \frac{999}{92800}x_2^2x_3 + \frac{13173}{371200}x_2^2x_4 \\
&- \frac{749}{2320}x_1x_3^2 + \frac{1273}{2320}x_2x_3^2 + \frac{637}{232}x_3^2x_4 - \frac{5}{928}x_1x_4^2 \\
&+ \frac{635}{928}x_2x_4^2 + \frac{415}{116}x_3x_4^2 + \frac{413}{46400}x_1x_2x_3 \\
&+ \frac{5911}{185600}x_1x_2x_4 - \frac{1443}{4640}x_1x_3x_4 + \frac{5751}{4640}x_2x_3x_4.
\end{aligned}
$$