LINEAR FLUTTER PREDICTION FOR A CANTILEVER WING MODEL

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Abstract
A simple cantilever wing was modeled using the Rayleigh-Ritz approach. Based on the quasi-steady and unsteady strip aerodynamics and the Lagrange equations, the equations are derived for mass, stiffness, and damping matrices of the wing model. Aerodynamics loads were considered in both constant and oscillatory basis and more accurate expressions are introduced for the Theodorsen functions. Flutter behavior of the wing model was analyzed using several combinations of bending and torsion modes. In overall, Flutter occurred at higher values of the wind speeds for the wing modeled with the frequency dependent aerodynamics.

Keywords: Quasi-Steady Aerodynamics – Wing Flutter – Frequency Dependent Aerodynamics.

Introduction

Flutter is the condition when an aircraft component (e.g. wing, control surface or tail-plane) exhibits self-sustained divergent oscillations (Dowell et al. [1]). The speed at which this occurs is known as the flutter speed. At speeds below the flutter speed, any initial dynamic structural vibrations will be damped, whereas at speeds above the flutter speed these disturbances will grow, leading to structural failure if not limited by aerodynamic or structural non-linearity. It is essential that all aircraft being designed such that flutter will not occur.

In addition to developing sophisticated structural, aerodynamic and aeroelastic mathematical models, a significant degree of testing on the ground and in the air is undertaken to demonstrate that flutter does not occur throughout the desired flight envelope (with a 20% safety margin). Ground vibration and flight flutter tests are mandatory in order to satisfy airworthiness regulations [2].

Having developed a mathematical model, the flutter stability of an aircraft is usually examined by calculating the eigenvalues of the state-space form of the aeroelasticity equation at different flight conditions. In this classical method, the frequency and damping ratio of each of the complex modes derived from the eigenvalues solution are plotted against airspeed (or Mach number). A zero damping value indicates the onset of flutter. For unsteady aerodynamics, the above method is more complicated because a trial and error (frequency matching [3]) scheme must be employed in order to obtain the frequency of the mode that causes flutter to occur.

A simple approach to obtain an estimate for the flutter frequency and the flutter speed of a binary flutter system based upon the Routh-Hurwitz [4] method has been known for many years. However, it has not been feasible to use it on larger systems or to include frequency dependent aerodynamic forces.

In this work, an approach based on the Rayleigh-Ritz model is developed to calculate the flutter speed and frequency of aeroelastic systems with greater than two degrees of freedom (DOF). According to this approach, the motion of a wing can be approximated by the summation of a finite number of mode shapes multiplied by the generalized co-ordinates.

The oscillatory aerodynamics forces can be expressed by the Theodorsen function $C(\nu)=F(\nu)+iG(\nu)$ which is a complex function of the reduced frequency ν as given in [3, 5-7] and is explored in more details follow

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Governing Equations of Motion Using Lagrangian Method

The equations of motion for a wing may be found using Lagrange equation [3] defined as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)+\frac{\partial F}{\partial q_i}-\frac{\partial T}{\partial q_i}+\frac{\partial U}{\partial q_i}=Q_i \qquad (1)$$

, where T is the kinetic energy, U is the potential energy, F is the work of dissipative force applied to the system, Q_i is the ith generalized force and q_i is the ith generalized coordinate.

Lagrange's equation is a powerful tool to apply to complex systems, as long as the generalized coordinates have been chosen correctly. This choice of coordinates is performed using the Rayleigh-Ritz method. To explain this, consider a simple cantilever wing model, i.e. a uniform flat plate with the thickness t, clamped in root and freely unsupported in tip as shown in the Figure 1.

A Cartesian co-ordinate system is used to describe the wing as follows:
X-axis: along the chord, origin at the wing root and at the leading edge.
Y-axis: vertical displacement, positive upward.
Z-axis: along the semi-span, from root to tip.

According to the Rayleigh-Ritz model, the motion of the wing can be approximated by the summation of a finite number of mode shapes multiplied by the generalized co-ordinates. Here, the bending and torsion modes for the wing are considered where can be approximated by

$$h=\sum_{i=1}^{m} z^{i+1} q_i \qquad \theta=\sum_{i=1}^{n} z^{i} q_{m+i} \qquad (2)$$

where h is the displacement of the wing tip in Y-direction. The mode shape $h=zq_1$ is not considered because the boundary condition at the root (cantilever) requires that the slope of the wing is zero there. θ is the rotation of the wing tip along the Z-axis, m is the total number of bending modes, and n is the total number of torsion modes.

Based on the quasi-steady strip theory and the assumptions were made by Dimitriadis [3] for evaluating the system of equations using the Lagrange's equations, the equations are derived for the mass, stiffness, and damping matrices of the wing model as follow.

Having modeled the wing and its motion, it is now straightforward to obtain the mass and structural stiffness matrices using Lagrange's equation. For this the kinetic and potential energy have to be calculated. First the Y-coordinate for every point on the wing is obtained as follows (see Figure 2):

$$y(x)=h+h_\theta(x) \qquad (3)$$

Therefore $h_\theta(x)$ has to be calculated. Assuming that θ << 1:

$$h_\theta(x)=\theta\left(x_f-x\right)=\sum_{i=1}^{n} z^i q_{m+i}\left(x_f-x\right) \qquad (4)$$

, and

$$y(x)=\sum_{i=1}^{m} z^{i+1} q_i+\sum_{i=1}^{n} z^i q_{m+i}\left(x_f-x\right) \qquad (5)$$

where x_f is the X-coordinate of the flexural axis. Note that with this model, the chord is always a straight line. In order to obtain a curved chord, powers of $\left(x_f-x\right)$ have to be used.

The kinetic energy for the beam system is obtained by

$$T=\frac{1}{2}\int_0^L\int_0^c \gamma_m \dot{y}^2\,dxdz$$
$$=\frac{1}{2}\int_0^L\int_0^c \gamma_m\left(\sum_{i=1}^{m} z^{i+1}\dot{q}_i+\sum_{i=1}^{n} z^i \dot{q}_{m+i}\left(x_f-x\right)\right)^2 dxdz \qquad (6)$$

, where γ_m is the wing mass per unit area. Inserting into the term $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)$ in Lagrange equation, it can be seen that there are no terms including q_i and therefore, the following term in Lagrange's equation must be equals to zero.

$$\left(\frac{\partial T}{\partial q_i}\right)=0 \qquad (7)$$

Moreover, the potential energy U, is defined as:

$$U=\frac{1}{2}\int_0^L GJ\left(\frac{\partial\theta}{\partial z}\right)^2 dz+\frac{1}{2}\int_0^L EI\left(\frac{\partial^2 h}{\partial z^2}\right)^2 dz$$
$$=\frac{1}{2}\int_0^L GJ\left(\sum_{i=1}^{n} z^{i-1} q_{m+i}\right)^2 dz+\frac{1}{2}\int_0^L EI\left(\sum_{i=1}^{m} z^{i-1} q_i\right)^2 dz \qquad (8)$$

which is the integral over the span, of the sum of the torsional and bending potential energies of a strip of wing. Equation 8 is implemented in Lagrange's equation by evaluating $\partial U/\partial q_i$. The work of the dissipative force applied to the system, F, is assumed to be zero.

Therefore, the only term in Lagrange's equation which has not yet been defined is Q_i, the ith generalised force which is given by:

$$Q_i = \frac{\partial(\delta W)}{\partial(\delta q_i)} \tag{9}$$

where δW is the work done by the forces applied on the system when the wing moves by incremental distances δy and/or δθ.

The forces applied on the system are the aerodynamic lift, drag, and moment. Since the wing is assumed to be rigid in the x-direction the drag does not produce any work and therefore is not considered. The aerodynamic moment produces work when the wing twists in the θ- direction.

The elements of the mass, stiffness, and damping matrices of the wing model suitable for programming are summarized in the appendix.

Unsteady Strip Theory for Oscillatory Aerodynamics Loads

The quasi-steady aerodynamics assumes that the aerodynamic forces on a body such as an aerofoil change instantaneously when the attitude of the body (heave and/or pitch in the case of an aerofoil) changes. However in reality, it takes time for the circulation round the body to change and therefore, if the body is constantly changing attitude as is the case in an oscillatory motion, the aerodynamic forces never reach their quasi-steady values. Additionally, apart from the circulatory aerodynamic forces (lift and moment) extra non-circulatory terms have to be added because, as the body moves, it carries with it an apparent mass, i.e. an adjacent cylindrical mass of air which acts as a reactive force on the body.

Unsteady strip theory assumes that the change of the aerodynamic forces around the body due to a change in attitude can be modelled by Wagner's function. Therefore, the lift on a strip of wing can be given by (reference 11):

$$dL = \frac{1}{2}\rho V^2 \left[L_z\left(\frac{y}{c}\right) + L_{\dot{z}}\left(\frac{\dot{y}}{V}\right) + L_\theta \theta + L_{\dot{\theta}}\left(\frac{\dot{\theta} c}{V}\right)\right] dz \tag{10}$$

and the moment is given by:

$$dM = \frac{1}{2}\rho V^2 c^2 \left[M_z\left(\frac{y}{c}\right) + M_{\dot{z}}\left(\frac{\dot{y}}{V}\right) + M_\theta \theta + M_{\dot{\theta}}\left(\frac{\dot{\theta} c}{V}\right)\right] dz \tag{11}$$

where c is the wing chord, and V is the air or wing speed. L_z, L_θ,...and M_z, M_θ,... are called the oscillatory aerodynamic derivatives. These are non-dimensional parameters which depend upon the frequency and therefore incorporate the unsteadiness of the motion and are defined as follows [3]:

$$L_z = 2\pi\left[\frac{-\nu^2}{4} - \nu G(\nu/2)\right]$$

$$L_{\dot{z}} = 2\pi F(\nu/2)$$

$$L_\theta = 2\pi\left[-\frac{\nu^2}{4}\left(e-\frac{1}{4}\right) + F(\nu/2) - \nu G(\nu/2)\left(\frac{1}{2}-e\right)\right]$$

$$L_{\dot{\theta}} = 2\pi\left[\frac{1}{4} + F(\nu/2)\left(\frac{1}{2}-e\right) + \frac{G(\nu/2)}{\nu}\right] \tag{12}$$

where e is the normalized distance between the flexural axis and aerodynamic centre with respect to the chord length. The functions $F(\nu)$ and $G(\nu)$ are the components of the Theodorsen function $C(\nu)=F(\nu)+iG(\nu)$ and their corresponding numerical values are found in reference [3]. The Theodorsen function, $C(\nu)$, is an incorporation of Wagner's function and depends upon ν which is a frequency parameter and is given by $\nu=\omega c/V$ (ω is the frequency of the oscillation).

In the same way the moment derivatives are given by:

$$M_z = 2\pi\left[\frac{-\nu^2}{4}\left(e-\frac{1}{4}\right) - e\nu G(\nu/2)\right]$$

$$M_{\dot{z}} = 2\pi\left[eF(\nu/2)\right]$$

$$M_\theta = 2\pi\left[\frac{\nu^2}{4}\left(\left(e-\frac{1}{4}\right)^2 + \frac{1}{32}\right) + eF(\nu/2) - e\nu G(\nu/2)\left(\frac{1}{2}-e\right)\right]$$

$$M_{\dot{\theta}} = 2\pi\left[-\frac{1}{4}\left(\frac{1}{2}-e\right) + eF(\nu/2)\left(\frac{1}{2}-e\right) + e\frac{G(\nu/2)}{\nu}\right] \tag{13}$$

Analytical expressions were used in [3] to represent approximation to the Theodorsen function; however, up to 10 percent error was unavoidable. In this work, far better curve fitting functions are introduced by less than 1 percent error as follow:

$$F(\nu) = \begin{cases} \dfrac{1.018203 + 2.2869727\nu}{1 + 5.1617432\nu - 0.059354252\nu^2} & 0 \le \nu \le 10 \\ 0.5 & \nu > 10 \end{cases}$$

$$-G(\nu) = \begin{cases} \dfrac{0.005049948 + 4.2431515\nu}{1 + 11.713961\nu + 29.445835\nu^2} & 0 \le \nu \le 10 \\ 0 & \nu > 10 \end{cases} \tag{14}$$

, which are compared with theoretical values of $F(\nu)$and $G(\nu)$ in Figures 3 and 4, respectively.

Solution of Aeroelastic Equation of Motion

Assembling together all terms of the Lagrange's equation gives equation of motion for the wing in terms of mass, M, aerodynamic damping, D, aerodynamics stiffness, K_{aero}, and structural stiffness, K_{struct}, as below:

$$M\ddot{q} + D\dot{q} + \left(K_{aero} + K_{struct}\right)q = 0 \qquad (15)$$

For the sake of simplicity, the above equation may be written with a unique stiffness matrix as:

$$M\ddot{q} + D\dot{q} + Kq = 0 \qquad (16)$$

By multiplying by the inverse of the mass matrix, the equation (16) can be rewritten in the state-space form to give:

$$\begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix}\begin{bmatrix} \ddot{q} \\ \dot{q} \end{bmatrix} + \begin{bmatrix} -M^{-1}D & -M^{-1}K \\ -I & 0 \end{bmatrix}\begin{bmatrix} \dot{q} \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (17)$$

, which can be rewritten as:

$$-\dot{p} + Qp = 0$$
$$p = \begin{bmatrix} \dot{q} \\ q \end{bmatrix}, \quad Q = \begin{bmatrix} -M^{-1}D & -M^{-1}K \\ -I & 0 \end{bmatrix} \qquad (18)$$

This can be expressed is in the form of the classical eigenvalue problem by setting $p = Pe^{\lambda t}$ to obtain

$$\left[-\lambda I + Q\right]Pe^{\lambda t} = 0 \qquad (19)$$

By solving the above for λ the following general solution is obtained:

$$\lambda_{1,2,\cdots,2(m+n)-1,2(m+n)} = -\xi\omega \pm i\omega\sqrt{1-\xi^2} \qquad (20)$$

where ξ is the damping ratio and ω is the natural frequency of the motion. The natural frequency is the frequency of oscillation when the damping is assumed to be zero. A number of very important pieces of information can be obtained from these two parameters. The damping ratio, especially, determines whether the wing is going to flutter or not. Since the damping ratio is a measure of the damping present in the actual motion, if it is positive (i.e. if the real part of λ is negative) the oscillation will die down eventually. Conversely, if the damping ratio is negative (i.e. the real part of λ is positive), the oscillation will grow and flutter will occur.

The frequency of the mode is useful in estimating exactly how stable or unstable the motion is and in determining the type of flutter that the wing will undergo. It will prove a lot more useful in evaluating unsteady strip theory when frequency matching ought to be performed.

Having found the eigenvalues, it remains to obtain the eigenvectors. This is done by inserting the eigenvalues back into equation (19) and then finding the vectors P. The eigenvectors are useful in determining the mode shapes and hence the importance of each type of mode (bending or torsion). The generalized coordinates are no more than measures of how much of each mode is present in the motion. The eigenvectors, like the eigenvalues, come in pairs of complex conjugate vectors, one pair for each mode. However, they are the solutions to equation (19) and not the original generalized coordinates q_1, q_2, ..., q_{m+n}. From equation (19) and the definition of P, the generalized coordinates themselves are determined by

$$q_{1,\cdots,m+n} = p_{m+n+1,\cdots,2(m+n)} \qquad (21)$$

Since the eigenvectors are complex, each eigenvector has a phase angle and amplitude. Only the amplitudes are used. This gives a mode shape indicating the maximum bending (h) or torsional (θ) deflection of the wing at various points along the span of the wing. If the eigenvectors are normalized by the maximum element of the vector then the various modes' contributions can be compared to each other.

Results and Discussions

The cantilevered wing model with the numerical specifications given in Table 1 was considered to investigate the effects of oscillatory aerodynamics on the linear flutter speed. Note that the $M_{\theta\cdot}$ value given in Table 1 was used when non-oscillatory aerodynamics considered. However, for the frequency dependent aerodynamics, the $M_{\theta\cdot}$ value was determined from the equation (13).

Calculations were carried out by programming routines in *MATLAB*. The particular difficulty encountered was the sorting of the eigenvalues. The eigenvalues are generated by the relevant *MATLAB* routine in the form of a diagonal matrix in increasing order of their modulus. This caused the respective eigenvalues for each mode to be mixed up. There is no satisfactory simple solution for this problem especially in the case of a multi-bending, multi-torsion motion and therefore an easy, though not tremendously sophisticated approach was followed. The eigenvalues were split into two column vectors, one containing the real and the other the imaginary parts. Then, the new column vectors were sorted separately in increasing order. This did not necessarily give the correct eigenvalue for each

particular mode but at least it produced continuous graphs of damping ratio versus velocity and natural frequency versus velocity. The rest of the required sorting was done manually.

As the first example, the two degrees of freedom cantilever wing model with constant quasi-steady aerodynamics is considered. The wing specification was set according to the values given in Table 1.

The results including the damping ratio and the natural frequency are plotted against airspeed in the Figure 4. As it seen, the wing flutters at its bending mode where the damping ratio becomes zero. The flutter speed and corresponding frequency value of $V_F = 94.0774 \quad (m/s), \quad \omega_F = 110.208 \quad (rad/s)$ were obtained, respectively. The damping drops drastically once flutter occurs which suggest a divergence flutter. The torsional mode reappears after it becomes zero and eventually assumes the value of the bending natural frequency. This becomes more evident when the damping ratio curves are studied.

By increasing the number of bending modes to two, the flutter speed was slightly reduced and the corresponding natural frequency increased to the values of $V_F = 93.9415 \quad (m/s), \quad \omega_F = 111.228 \quad (rad/s),$ respectively, as shown in Figure 5. Again, the same features can be observed and it appeared that the first bending mode is still critical.

In Figure 6 by increasing torsional modes into 3, higher frequencies appeared and damping ratio corresponding the torsional modes get closer to zero value; however, they remain almost constant throughout the range of airspeed shown.

For frequency dependent aerodynamics, the approach introduced by Sedaghat [8, 9] was adopted to find the flutter speed, V_F, and its corresponding frequency parameter, ν_F. This value of frequency was used to determine the oscillatory aerodynamics. The computational exercise was repeated for the three cases explained above.

As shown in Figure 7 for the wing with one bending mode and one torsional mode, the flutter speed $V_F = 105.444 \ (m/s)$ and the frequency parameter of $\nu_F = 0.244619$ are obtained which from the natural frequency curve the value of $\omega_F = 85.9788 \ (rad/s)$ can be read at flutter speed. The pitching moment was determined to be $M_{\dot{\theta}}(\nu_F) = -1.10584$ from the equation (13) introduced. The same features as the Figure 5 can be observed in Figure 8 except that the flutter speed increased and its corresponding natural frequency decreased when including oscillatory aerodynamic terms.

In Figure 9, the results are fairly comparable with Figure 6. The flutter speed of $V_F = 105.424 \ (m/s)$ and the frequency parameter of $\nu_F = 0.245$ are obtained [8, 9] deducing the natural frequency value of $\omega_F = 86.13 \ (rad/s)$ at flutter speed. The pitching moment was determined to be $M_{\dot{\theta}}(\nu_F) = -1.1047$.

For the five degrees of freedom wing model, the results are plotted in Figure 10. The results have not considerably changed from the three degrees of freedom wing model shown in Figure 9 and explained above. In overall, same features as the Figure 5 can be observed in Figure 8 except that the flutter speed increased and its corresponding natural frequency decreased when including oscillatory aerodynamic terms. Flutter occurred at higher wind speeds for the wing modeled with frequency dependent aerodynamics.

Conclusions

The cantilevered wing model was considered to investigate the effects of oscillatory aerodynamics on the linear flutter speed. Aeroelastic equation of motion was derived and solution to the corresponding eigenvalue problem was obtained in the form of the damping ratios and the natural frequencies. The computations were carried out using programmed routines in *MATLAB* for both constant and oscillatory aerodynamics. For oscillatory aerodynamic, the symbolic approach introduced in [8, 9] was used to find directly the exact frequency parameter at flutter.

The wing with the following three configurations was investigated to estimate the flutter speed and its natural frequency:

1. One bending and one torsional modes
2. Two bending and one torsional modes
3. Two bending and three torsional modes

It is found that the bending modes are the critical modes and they first have reached to the flutter condition. Moreover, oscillatory aerodynamics terms added to the modeling has increased the performance of wing by increasing the linear flutter speed. However, their corresponding natural frequencies were decreased. These changes are of great importance when flight flutter tests are performed and/or modern aircrafts are being designed.

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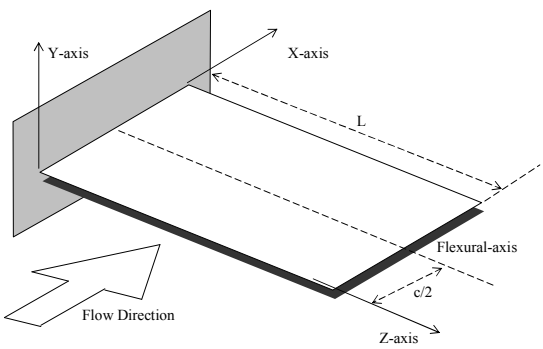

Fig. 1- A cantilevered rectangular wing model.

Table 1- Cantilevered wing model specifications

Parameter	Numerical Value	Formula and/or unit
L	2	m
ρ *(air)*	1.225	Kg/m^3
c	0.3	m
t	0.01	m
I	25×10^{-9}	$t^3c/12\ \left(m^4\right)$
E	70×10^{9}	Pa
J	10^{-7}	$t^3c/3\ \left(m^4\right)$
G	26×10^{9}	Pa
ρ_m	2.7×10^{3}	kg/m
m	270	$\rho_m t\left(kg\right)$
a	2π	
x_f	0.12	$0.4c\ \left(m\right)$
x_m	0.15	$0.5c\ \left(m\right)$
x_a	0.075	$0.25c\ \left(m\right)$
e	0.15	$\left(x_f - x_a\right)/c$
$M_{\dot{\theta}}$	-0.22	$\pi\left(6e-8e^2-1\right)/4$

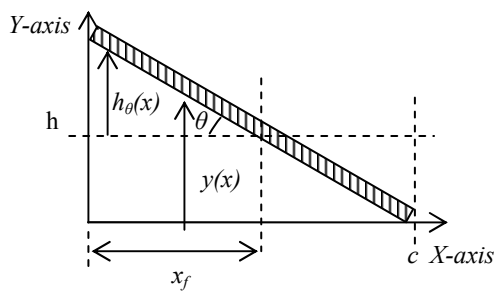

Fig. 2- Side view of the deformed wing

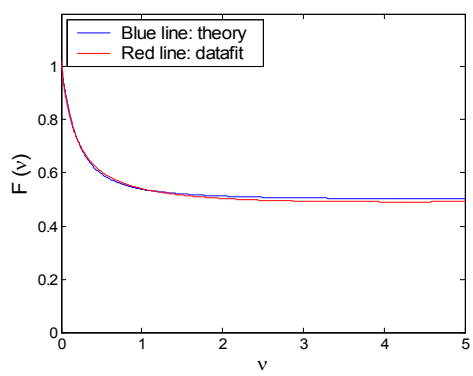

Fig. 3- Data fit for the Theodorsen function F(v).

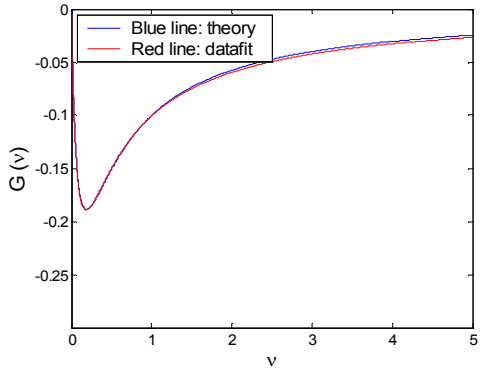

Fig. 4- Data fit for the Theodorsen function G(ν).

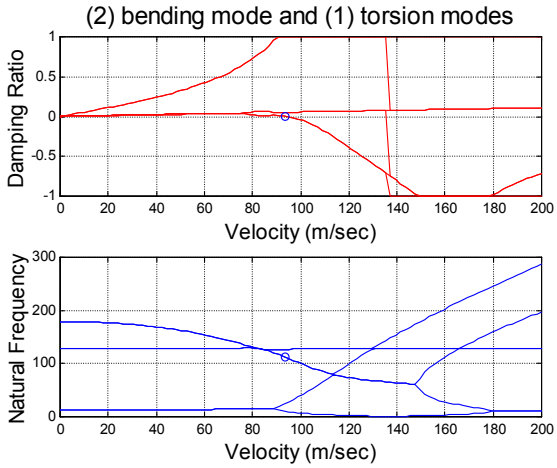

Fig. 5- The two degrees of freedom cantilever wing model with constant quasi-steady aerodynamics
(V_F = 94.0774 (m/s), $\omega_F = 110.208$ (rad/s)).

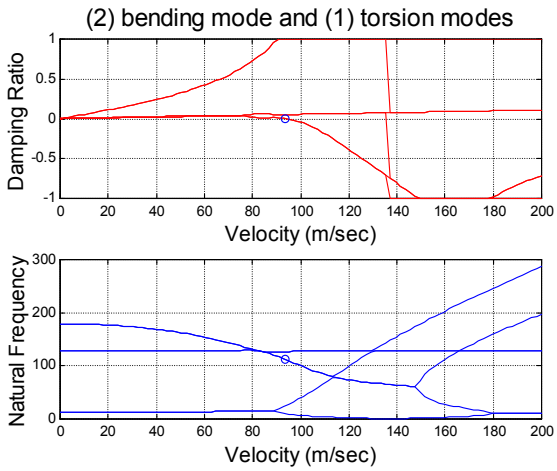

Fig. 6- The three degrees of freedom cantilever wing model with constant quasi-steady aerodynamics
(V_F = 93.9415 (m/s), $\omega_F = 111.228$ (rad/s)).

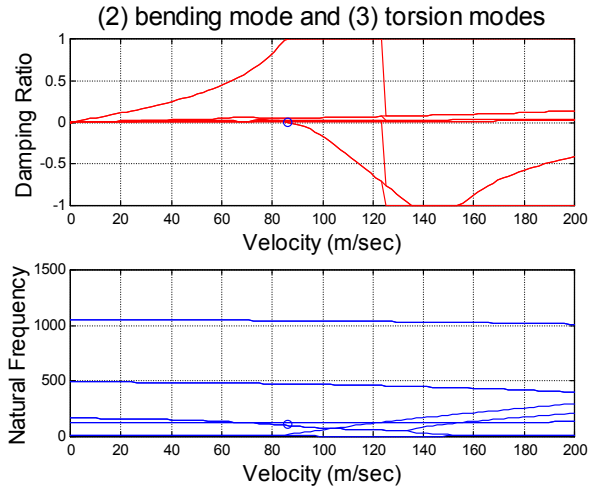

Fig. 7- The five degrees of freedom cantilever wing model with constant quasi-steady aerodynamics
(V_F = 86.5602 (m/s), $\omega_F = 99.7136$ (rad/s)).

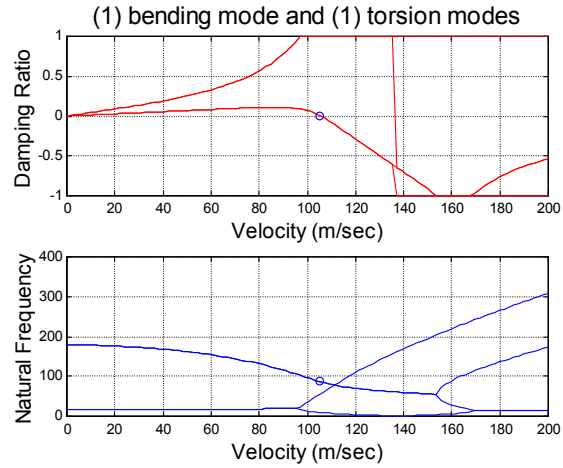

Fig. 8- The two degrees of freedom cantilever wing model with frequency dependent aerodynamics.

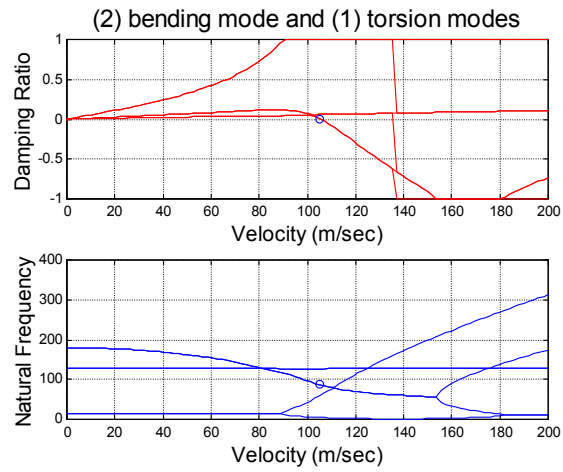

Fig. 9- The three degrees of freedom cantilever wing model with frequency dependent aerodynamics.

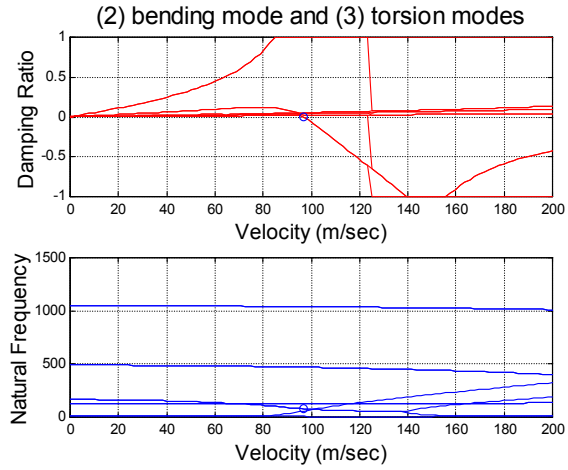

Fig. 10- The five degrees of freedom cantilever wing model with frequency dependent aerodynamics.

Appendix

Expressions are derived based on the following intervals:

$$I_1 = i \in [1,m], \quad j \in [1,m]$$
$$I_2 = i \in [1,m], \quad j \in [m+1,m+n]$$
$$I_3 = i \in [m+1,m+n], \quad j \in [1,m]$$
$$I_4 = i \in [m+1,m+n], \quad j \in [m+1,m+n]$$

for the mass, stiffness, and damping matrices of the $(m \times n)$ D.O.F wing model given below.

Mass:

$$M_{ij} = \frac{\rho_m . L^{(2m+5-i-j)} . c}{(2m+5-i-j)}, \quad \text{on } I_1$$

$$M_{ij} = \frac{-\rho_m . L^{(3-i+j)} . ((xf-c)^2 - xf^2)}{2 \times (3-i+j)}, \text{ on } I_2$$

$$M_{ij} = \frac{-\rho_m . L^{(3+i-j)} . ((xf-c)^2 - xf^2)}{2 \times (3+i-j)}, \text{ on } I_3$$

$$M_{ij} = \frac{-\rho_m . L^{(i+j-2m+1)} . ((xf-c)^3 - xf^3)}{3 \times (i+j-2m+1)}, \text{ on } I_4$$

Damping:

$$D_{ij} = \frac{\rho \mathrm{VaL}^{(2m+5-i-j)} c}{2(2m+5-i-j)}, \text{ on } I_1$$

$$D_{ij} = \frac{\rho \mathrm{VaL}^{(3-i+j)} ec^2}{2(3-i+j)}, \text{ on } I_2$$

$$D_{ij} = \frac{\rho \mathrm{VaL}^{(3+i-j)} ec^2}{(3+i-j)}, \text{ on } I_3$$

$$D_{ij} = \frac{\rho \mathrm{VaL}^{(i+j-2m+1)} e^2 c^3}{(i+j-2m+1)} - \frac{M_{\dot{\theta}} \rho \mathrm{Vc}^3 \mathrm{L}^{(i+j-2m+1)}}{2(i+j-2m+1)}, \text{ on } I_4$$

Stiffness:

$$K_{ij} = \left(\frac{\mathrm{EIL}^{(2m+1-i-j)} (m-i+2)}{(2m+1-i-j)} \times \frac{(m+1-i)(m-j+2)(m+1-j)}{1} \right), \text{ on } I_1$$

$$K_{ij} = \frac{-\rho \mathrm{V}^2 acL^{(3-i+j)}}{2(3-i+j)}, \text{ on } I_2$$

$$K_{ij} = 0, \text{ on } I_3$$

$$K_{ij} = \left(\frac{\mathrm{GJL}^{(i+j-1-2m)} (i-m)(j-m)}{(i+j-1-2m)} - \frac{\rho \mathrm{V}^2 aec^2 \mathrm{L}^{(i+j-2m+1)}}{(i+j-2m+1)} \right), \text{ on } I_4$$

where L is the length of wing or the semi-span, ρ is the air density, ρ_m is the wing mass per thickness, t is the wing thickness, a is the lift curve slope, e is the normalized distance between the flexural axis and aerodynamic centre with respect to the chord length, c is the wing chord, V is the air or wing speed, $M_{\dot{\theta}}$ is aerodynamic moment torsional damping derivative, E is the Young's Modulus, I is the area moment of inertia, G is the Modulus of rigidity, and J is the polar moment of inertia.