COMPOUND CURVATURE IN CYLINDRICAL POLARS

CONSIDER A SURFACE $z = \zeta(r,\theta)$ IN CYLINDRICAL COORDINATES (r,θ,z). WE COMPUTE GENERAL EXPRESSIONS FOR THE OUTWARD NORMAL $\bar{m}$ AND THE COMPOUND CURVATURE $K(r,\theta)$

(a) IN THE LIMIT OF LINEAR DISTURBANCES $\zeta = \epsilon \zeta'$ WE SHOW THAT APPROXIMATION TO THE COMPOUND CURVATURE, SHOWING THAT IT REDUCES TO A TWO-DIMENSIONAL LAPLACIAN.

(b) FOR AXISYMMETRIC DISTURBANCES $z = z(r)$ ONLY. WE FIND THE COMPOUND CURVATURE $K(z(r))$ IN THIS CASE AND SHOW THAT ONE CAN IDENTIFY EXPRESSIONS FOR THE PRINCIPAL RADII OF CURVATURE R_1 AND R_2 IN THE (r,θ) AND (r,z) COORDINATE PLANES.

SOMETIMES IT IS MORE CONVENIENT TO WORK WITH $r = r(z)$. USING A SWITCHING OF INDEPENDENT AND DEPENDENT VARIABLES

$$\frac{dz}{dr} = \frac{1}{dr/dz} \qquad \frac{d^2z}{dr^2} = -\frac{1}{(dr/dz)^3}\,\frac{d^2r}{dz^2}$$

ONE FINDS A NEW EXPRESSION FOR COMPOUND CURVATURE $K(r(z))$, AGAIN WITH IDENTIFICATION OF PRINCIPLE RADII OF CURVATURE R_1 AND R_2 IN THE (r,θ) AND (r,z) COORDINATE PLANES.

IN CYLINDRICAL COORDINATES

$$\bar{m} = \frac{\nabla F}{|\nabla F|} \qquad \nabla = \bar{e}_r \frac{\partial}{\partial r} + \bar{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \bar{e}_z \frac{\partial}{\partial z}$$

$$F(r,\theta,z,t) = \zeta(r,\theta,t) - z = 0$$

THEN

$$\frac{\nabla F}{|\nabla F|} = \frac{F_r \bar{e}_r + \frac{1}{r} F_\theta \bar{e}_\theta + F_z \bar{e}_z}{\pm \sqrt{F_r^2 + \frac{1}{r^2} F_\theta^2 + F_z^2}}$$

NOW $F_r = \zeta_r \quad F_\theta = \zeta_\theta \quad F_z = -1$

CHOOSE NEGATIVE SIGN SO $\bar{m}$ ALIGNED WITH $\bar{e}_z$

$$\boxed{\bar{m} = \frac{-\zeta_r \bar{e}_r - \frac{1}{r}\zeta_\theta \bar{e}_\theta + \bar{e}_z}{\sqrt{\zeta_r^2 + \frac{1}{r^2}\zeta_\theta^2 + 1}}}$$

NOW IN CYLINDRICAL COORDINATES

$$\nabla \cdot \bar{A} = \frac{1}{J}\left[\frac{\partial}{\partial r}\left(\frac{J}{h_r} A_r\right) + \frac{\partial}{\partial \theta}\left(\frac{J}{h_\theta} A_\theta\right) + \frac{\partial}{\partial z}\left(\frac{J}{h_z} A_z\right)\right]$$

WHERE THE METRICAL COEFFICIENTS ARE

$$h_r = 1 \qquad h_\theta = r \qquad h_z = 1$$

AND THE JACOBIAN IS $J = r$; THUS

$$\nabla \cdot \bar{m} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r m_r) + \frac{\partial}{\partial \theta}(m_\theta) + \frac{\partial}{\partial z}(r m_z)\right]$$

NOW FOR $D = \sqrt{\zeta_r^2 + \frac{1}{r^2}\zeta_\theta^2 + 1}$

$$\frac{\partial}{\partial r}(r m_r) = \frac{\partial}{\partial r}\left(\frac{-r\zeta_r}{D}\right)$$

$$= -\left[\frac{D(r\zeta_r)_r - r\zeta_r\left(\frac{1}{\cancel{2}D}\left\{\cancel{2}\zeta_r\zeta_{rr} + \frac{\cancel{2}}{r^2}\zeta_\theta\zeta_{r\theta} - \frac{\cancel{2}}{r^3}\zeta_\theta^2\right\}\right)}{D^2}\right]$$

$$= \left[\frac{\left(\zeta_r^2 + \frac{1}{r^2}\zeta_\theta^2 + 1\right)(r\zeta_r)_r - r\zeta_r\left(\zeta_r\zeta_{rr} + \frac{1}{r^2}\zeta_\theta\zeta_{r\theta} - \frac{1}{r^3}\zeta_\theta^2\right)}{D^3}\right]$$

$$\frac{\partial}{\partial r}(r m_r) = \frac{-1}{D^3}\left[\zeta_r^2\left(r\cancel{\zeta_{rr}} + \zeta_r\right) + (r\zeta_r)_r\left(1 + \frac{\zeta_\theta^2}{r^2}\right) - r\cancel{\zeta_r^2}\zeta_{rr} - \frac{1}{r}\zeta_r\zeta_\theta\zeta_{r\theta} + \frac{1}{r^2}\zeta_r\zeta_\theta^2\right]$$

HENCE

$$\frac{\partial}{\partial r}(r m_r) = \frac{-1}{D^3}\left[\zeta_r^3 + (r\zeta_r)_r\left(1+\frac{\zeta_\theta^2}{r^2}\right) - \frac{1}{r}\zeta_r\zeta_\theta\zeta_{r\theta} + \frac{1}{r^2}\zeta_r\zeta_\theta^2 \right]$$

$$= \frac{-1}{D^3}\left[\zeta_r^3 + (r\zeta_r)_r + \left\{ r\zeta_{rr} + \zeta_r \right\}\frac{\zeta_\theta^2}{r^2} - \frac{1}{r}\zeta_r\zeta_\theta\zeta_{r\theta} + \frac{1}{r^2}\zeta_r\zeta_\theta^2 \right]$$

$$= \frac{-1}{D^3}\left[\zeta_r^3 + (r\zeta_r)_r + r\zeta_{rr}\frac{\zeta_\theta^2}{r^2} + \frac{2}{r^2}\zeta_r\zeta_\theta^2 - \frac{1}{r}\zeta_r\zeta_\theta\zeta_{r\theta} \right]$$

AND

$$\frac{\partial}{\partial\theta}(m_\theta) = \frac{\partial}{\partial\theta}\left[-\frac{1/r\,\zeta_\theta}{D} \right] = -\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\zeta_\theta}{D}\right)$$

$$= -\frac{1}{r}\left[\frac{D\zeta_{\theta\theta} - \zeta_\theta\frac{1}{\cancel{2}D}\left\{ \cancel{2}\zeta_r\zeta_{r\theta} + \frac{\cancel{2}}{r^2}\zeta_\theta\zeta_{\theta\theta} \right\}}{D^2} \right]$$

$$= -\frac{1}{r}\left[\frac{\left(\zeta_r^2 + \cancel{\frac{1}{r^2}\zeta_\theta^2} + 1\right)\zeta_{\theta\theta} - \zeta_\theta\left(\zeta_r\zeta_{r\theta} + \cancel{\frac{1}{r^2}\zeta_\theta\zeta_{\theta\theta}}\right)}{D^3} \right]$$

$$= -\frac{1}{rD^3}\left[\zeta_r^2\zeta_{\theta\theta} + \zeta_{\theta\theta} - \zeta_r\zeta_\theta\zeta_{r\theta} \right]$$

FINALLY

$$\frac{\partial}{\partial z}(r m_z) = r\frac{\partial m_z}{\partial z} = 0 \qquad \left\{ \begin{array}{l} \text{SINCE } \zeta = \zeta(r,\theta,t) \text{ ONLY} \\ \text{AND } \frac{\partial r}{\partial z} = 0 \end{array} \right\}$$

THUS

$$\pm D^3\nabla\cdot\bar{m} = \frac{1}{r}\left[\zeta_r^3 + (r\zeta_r)_r + \frac{1}{r}\zeta_{rr}\zeta_\theta^2 + \frac{2}{r^2}\zeta_r\zeta_\theta^2 - \frac{1}{r}\zeta_r\zeta_\theta\zeta_{r\theta} + \zeta_r^2\frac{\zeta_{\theta\theta}}{r} + \frac{\zeta_{\theta\theta}}{r} - \frac{1}{r}\zeta_r\zeta_\theta\zeta_{r\theta} \right]$$

AND THEN:

$$\boxed{\nabla\cdot\bar{m} = \frac{\frac{1}{r}\left[\zeta_r^3 + (r\zeta_r)_r + \frac{\zeta_{rr}\zeta_\theta^2}{r} + \frac{2}{r^2}\zeta_r\zeta_\theta^2 - \frac{2}{r}\zeta_r\zeta_\theta\zeta_{r\theta} + \frac{\zeta_{\theta\theta}}{r}\left(\zeta_r^2 + 1\right) \right]}{\pm\left(1 + \zeta_r^2 + \frac{1}{r^2}\zeta_\theta^2\right)^{3/2}}}$$

PART (a)

IN THE LIMIT OF LINEAR DISTURBANCES SET $\zeta = \epsilon \zeta'$ AND KEEP HIGHEST ORDER TERMS OF $O(\epsilon)$ ONLY; DROPPING PRIMES OBTAIN

$$K(r,\theta) = \frac{1}{r}\left[(r\zeta_r)_r + \frac{\zeta_{\theta\theta}}{r} \right]$$

$$\boxed{K(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \zeta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \zeta}{\partial \theta^2}}$$

TWO-DIMENSIONAL LAPLACIAN

PART (b) AXISYMMETRIC DISTURBANCES $\zeta = \zeta(r)$

$$\nabla \cdot \bar{m} = \frac{\frac{1}{r}\left[\zeta_r^3 + (r\zeta_r)_r \right]}{\pm (1+\zeta_r^2)^{3/2}}$$

THUS

$$\pm \nabla \cdot \bar{m} = \frac{1}{r}\frac{1}{(1+\zeta_r^2)^{3/2}}\left[\zeta_r^3 + r\zeta_{rr} + \zeta_r \right]$$

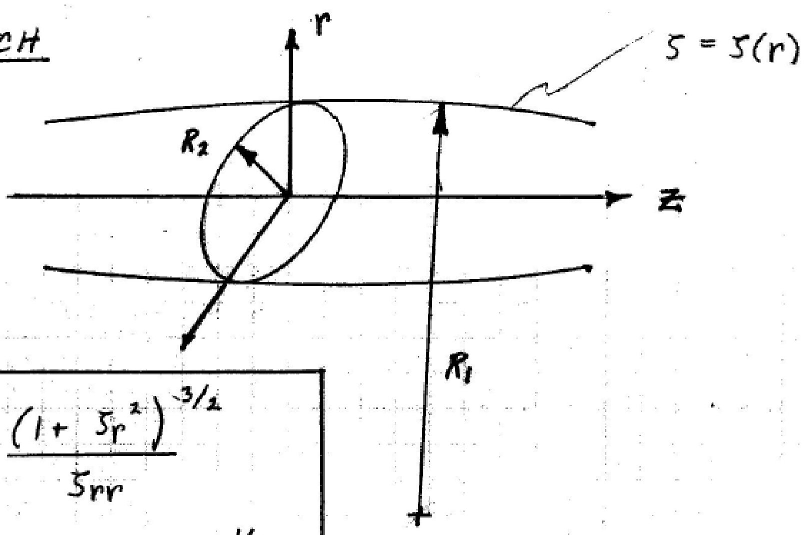
$$\pm \nabla \cdot \bar{m} = \frac{1}{r}\frac{1}{(1+\zeta_r^2)^{3/2}}\left[\zeta_r(1+\zeta_r^2) + r\zeta_{rr} \right]$$

$$\boxed{\pm \nabla \cdot \bar{m} = \frac{1}{r}\frac{\zeta_r}{(1+\zeta_r^2)^{1/2}} + \frac{\zeta_{rr}}{(1+\zeta_r^2)^{3/2}}}$$

NOW $\nabla \cdot \bar{m} = \frac{1}{R_1} + \frac{1}{R_2}$ WHERE R_1 AND R_2 ARE THE PRINCIPLE RADII OF CURVATURE.

CLEARLY $R_1 = (1+\zeta_r^2)^{3/2} / \zeta_{rr}$ IS THE CURVATURE IN THE r-z PLANE SO THE OTHER MUST BE THE CURVATURE IN THE r-θ PLANE.

SKETCH



$$R_1 = \frac{\left(1+\zeta_r^{\,2}\right)^{3/2}}{\zeta_{rr}}$$

$$R_2 = \frac{r\left(1+\zeta_r^{\,2}\right)^{1/2}}{\zeta_r}$$

PART (C)

NOW SWITCH DEPENDENT AND INDEPENDENT VARIABLES. WRITING $\zeta = z(r)$ WE NOW HAVE

$$\pm \nabla\cdot\bar{m} = \frac{1}{r}\,\frac{z_r}{(1+z_r^{\,2})^{1/2}} + \frac{z_{rr}}{(1+z_r^{\,2})^{3/2}}$$

WRITE $r = r(z)$; THEN

$$z_r = \frac{1}{\left(\frac{dr}{dz}\right)}$$

AND $$z_{rr} = \frac{d}{dr}\,\frac{1}{\left(\frac{dr}{dz}\right)} = \frac{d}{dz}\,\frac{1}{\left(\frac{dr}{dz}\right)}\cdot\frac{dz}{dr} = -\frac{1}{\left(\frac{dr}{dz}\right)^2}\,\frac{d^2r}{dz^2}\,\frac{1}{\left(\frac{dr}{dz}\right)}$$

THUS $$z_{rr} = -\frac{1}{\left(\frac{dr}{dz}\right)^3}\,\frac{d^2r}{dz^2}$$

OBTAIN

$$\pm\nabla\cdot\bar{m} = \frac{1}{r}\,\frac{1}{r_z}\,\frac{1}{\left[1+\frac{1}{(r_z)^2}\right]^{1/2}} - \frac{1}{(r_z)^3}\,\frac{r_{zz}}{\left[1+\frac{1}{(r_z)^2}\right]^{3/2}}$$

HENCE

$$\pm \nabla \cdot \bar{m} = \frac{1}{r} \frac{1}{(1+r_z^2)^{1/2}} - \frac{r_{zz}}{(1+r_z^2)^{3/2}}$$

NOW WRITING THE POSITION OF THE SURFACE AS

$$r = \zeta(z)$$

OBTAIN

$$\boxed{\pm \nabla \cdot \bar{m} = \frac{1}{r} \frac{1}{(1+\zeta_z^2)^{1/2}} - \frac{\zeta_{zz}}{(1+\zeta_z^2)^{3/2}}}$$

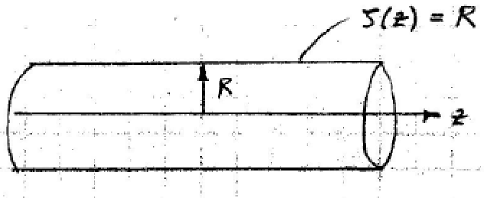
NOW THE PRINCIPAL RADII OF CURVATURE ARE

IN $r-z$ PLANE $\quad R_1 = -\dfrac{(1+\zeta_z^2)^{3/2}}{\zeta_{zz}}$

IN $r-\theta$ PLANE $\quad R_2 = r\,(1+\zeta_z^2)^{1/2}$

QUESTION: WHAT ARE R_1, R_2 FOR A CYLINDRICAL SURFACE $r = R$?

IN THIS CASE $\quad \zeta(z) = R = \text{CONST}$



$\therefore\ \zeta_z = 0;\ \zeta_{zz} = 0$

$R_1 = \infty$

$R_2 = R$